

Deterministic and Stochastic Differential Equations in Hilbert Spaces Involving Multivalued Maximal Monotone Operators*

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Abstract

This work deals with a Skorokhod problem driven by a maximal operator:

$$\begin{cases} du(t) + Au(t)(dt) \ni f(t)dt + dM(t), & 0 < t < T, \\ u(0) = u_0, \end{cases}$$

that is a multivalued deterministic differential equation with a singular inputs $dM(t)$, where $t \rightarrow M(t)$ is a continuous function. The existence and uniqueness result is used to study an Itô's stochastic differential equation

$$\begin{cases} du(t) + Au(t)(dt) \ni f(t, u(t))dt + B(t, u(t))dW(t), & 0 < t < T, \\ u(0) = u_0, \end{cases}$$

in a real Hilbert space H , where A is a multivalued (α) -maximal monotone operator on H , and $f(t, u)$ and $B(t, u)$ are Lipschitz continuous with respect to u . Some asymptotic properties in the stochastic case are also found.

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1 Introduction

Generally, the stochastic model for parabolic evolution systems with unilateral constraints (obstacle problem, one phase Stefan problem, Signorini problem) is an infinite dimensional stochastic differential equation of the form

$$\begin{cases} du(t) + Au(t) (dt) \ni f(t, u(t))dt + B(t, u(t))dW(t), & 0 < t < T, \\ u(0) = u_0, \end{cases} \quad (1)$$

where A is a maximal monotone operator in a Hilbert spaces H and $f(t, u)$ and $B(t, u)$ defined for $(t, u) \in [0, T] \times H$ are Lipschitz continuous with respect to u , and $\{W(t)\}_{t \geq 0}$ is a Wiener process with respect to a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$.

For finite dimensional case we mention the works of P.L. Lions & Sznitman [12], Y. Saisho [13] and the generalized result of E. Cipa [7]. The main idea is to consider a generalized Skorohod problem

$$\begin{cases} du(t) + Au(t) (dt) \ni f(t)dt + dM(t), & 0 < t < T, \\ u(0) = u_0, \end{cases} \quad (2)$$

and by continuity of Skorohod mapping

$$(u_0, f, M) \rightarrow u = S(u_0, f, M)$$

we obtain the existence and the uniqueness of the solution of equation (1). If in finite dimensional case one assume that $\text{int}D(A) \neq \emptyset$ and this assumption is essentially for the proof, in infinite dimensional case this assumption is too restrictive; it is not satisfied, not even for the obstacle problem. For this reason the step from finite to infinite dimensional case is not so directly.

V. Barbu and A. Răşcanu studied in [3] parabolic variational inequalities in the deterministic case, that is equations of the form (2) where $A = A_0 + \partial\varphi$. Some stochastic parabolic variational inequalities of the form (1) with $A = A_0 + \partial\varphi$ are considered by A. Bensoussan and A. Răşcanu in [5] and [4]. In this paper using the idea of looking the solution as image by a Skorohod mapping we prove the existence and the uniqueness of the solution of (1).

The paper is organized as follows. Some preliminaries deterministic results with a generalization of the solution for singular inputs are given in Section 2. Section 3 contains the main existence result on stochastic equation (1). Finally in Section 4 we give some asymptotic properties of the solution.

2 Deterministic evolution equations

2.1 Preliminaries

A. Throughout in this work H is a real separable Hilbert space with the norm $|\cdot|$ and the scalar product (\cdot, \cdot) , and $(X, \|\cdot\|_X)$ is a real separable Banach space with separable dual $(X^*, \|\cdot\|_{X^*})$. It is assumed that

$$X \subset H \cong H^* \subset X^*,$$

where the embedding are continuous with dense range. The duality paring (X^*, X) is denoted also (\cdot, \cdot) . Let $\gamma_0 > 0$ a constant of boundedness: $|\cdot|_H \leq \gamma_0 \|\cdot\|_X$.

B. If $[a, b]$ is a real closed interval and Y is a Banach space then $L^r(a, b; Y)$, $C([a, b]; Y)$, $BV([a, b]; Y)$, $AC([a, b]; Y)$, are the usual spaces of p -integrable, continuous, with bounded variation, and absolutely continuous Y -valued function on $[a, b]$, respectively. By $W^{1,p}([a, b]; Y)$ we shall denote the space of $y \in L^p(a, b; Y)$ such that $y' \in L^p(a, b; Y)$, where y' is the derivative in the sense of distributions. Equivalently (see e.g. [2], pag. 19, or [6]): $W^{1,p}([a, b]; Y) = \{y \in AC([a, b]; Y) : \frac{dy}{dt} \in L^p(a, b; Y); y(t) = y(a) + \int_a^t \frac{dy}{ds}(s)ds, \forall t \in [a, b]\}$. The space $W^{2,p}([a, b]; X)$ is similarly defined

C. A multivalued operator $A : H \rightarrow 2^H$ will be seen also as a subset of $H \times H$ setting for $A \subset H \times H$:

$$Ax = \{y \in H : [x, y] \in A\} \quad \text{and} \quad D(A) = \{x \in H : Ax \neq \emptyset\}.$$

The operator A is a maximal monotone operator if A is monotone i.e.

$$(y_1 - y_2, x_1 - x_2) \geq 0, \text{ for all } [x_1, y_1] \in A, [x_2, y_2] \in A$$

and it is maximal in the set of monotone operators: that is,

$$(v - y, u - x) \geq 0 \quad \forall [x, y] \in A, \quad \Rightarrow [u, v] \in A.$$

Let $\varepsilon > 0$ The following operators

$$J_\varepsilon x = (I + \varepsilon A)^{-1}(x) \text{ and } A_\varepsilon = \frac{1}{\varepsilon}(x - J_\varepsilon x),$$

are single-valued and they satisfy (see [2] and [6]) the properties for all $\varepsilon, \delta > 0$, $x, y \in H$:

- a) $[J_\varepsilon x, A_\varepsilon x] \in A$,
- b) $|J_\varepsilon x - J_\varepsilon y| \leq |x - y|$,
- c) $|A_\varepsilon x - A_\varepsilon y| \leq \frac{1}{\varepsilon} |x - y|$,
- d) $|J_\varepsilon x - J_\delta x| \leq |\varepsilon - \delta| |A_\delta x|$,
- e) $|J_\varepsilon x| \leq |x| + (1 + |\varepsilon - 1|) |J_1 0|$,
- f) $A_\varepsilon : H \rightarrow H$ is a maximal monotone operator.

(3)

Also

- a) $\overline{D(A)}$ is a convex set and $\lim_{\varepsilon \searrow 0} J_\varepsilon x = \Pr_{\overline{D(A)}} x, \forall x \in H,$
- b) $\forall [x, y] \in A: Ax$ is a closed convex set ,
- $$\lim_{\varepsilon \searrow 0} A_\varepsilon x = \Pr_{Ax} \{0\} \stackrel{def}{=} A^0 x \text{ and } |A_\varepsilon x| \leq |y|$$
- c) if $[x_n, y_n] \in A$ and
- $$\begin{aligned} & x_n \rightarrow x \text{ (strongly) in } H, y_n \xrightarrow{w} y \text{ (weakly) in } H, \text{ or} \\ & x_n \xrightarrow{w} x, \text{ and } y_n \rightarrow y, \text{ or} \\ & x_n \xrightarrow{w} x, y_n \xrightarrow{w} y, \quad \overline{\lim}_n (x_n, y_n) \leq (x, y) \end{aligned} \tag{4}$$
- then $[x, y] \in A,$
- d) if $\varepsilon_n \rightarrow 0, x_n \rightarrow x, A_{\varepsilon_n} y_n \xrightarrow{w} y$ then $[x, y] \in A$

Let $\alpha \in R$ be given. The operator $A : H \rightarrow 2^H$ is called α -maximal monotone operator if $A + \alpha I$ is the maximal monotone operator (I is the identity operator on H).

For $A : H \rightarrow 2^H$ an α -maximal monotone operator, $u_0 \in \overline{D(A)}, f \in L^1(0, T; H)$, the strong solution of the Cauchy problem

$$\frac{du}{dt} + Au \ni f_0(t), \quad a.e. t \in (0, T), \quad u(0) = u_0 \tag{5}$$

is defined as a function $u \in W^{1,1}([0, T]; H)$ satisfying $u(0) = u_0, u(t) \in D(A)$ a.e. $t \in (0, T)$, and there exists $h \in L^1(0, T)$, such that $h(t) \in Au(t)$ a.e. $t \in (0, T)$ and $du/dt + h(t) = f(t)$, a.e. $t \in (0, T)$. Such a solution is noted $u = S(u_0, f)$ and we remark that the strong solution is unique when this exists.

We recall from [1], p.31, that the following proposition holds :

Proposition 1 *If A is α -maximal monotone operator ($\alpha \in R$) on H , $u_0 \in D(A)$ and $f_0 \in W^{1,1}([0, T]; H)$ then the Cauchy problem (5) has a unique strong solution $u \in W^{1,\infty}([0, T]; H)$. Moreover if A_ε^α is the Yosida approximation of the operator $A + \alpha I$ and u_ε is the solution of the approximate equation*

$$\frac{du_\varepsilon}{dt} + A_\varepsilon^\alpha u_\varepsilon - \alpha u_\varepsilon = f, \quad u_\varepsilon(0) = u_0$$

then for all $[x_0, y_0] \in A$ there exists a constant $C = C(\alpha, T, x_0, y_0) > 0$ such that

$$c_1) \quad \|u_\varepsilon\|_{C([0, T]; H)}^2 \leq C \left(1 + |u_0|^2 + \|f\|_{L^1(0, T; H)}^2 \right) \text{ and}$$

$$c_2) \quad \lim_{\varepsilon \searrow 0} u_\varepsilon = u \text{ in } C([0, T]; H).$$

Here we shall study the equation (2) under the following basic assumptions:

$$(H_1) \left\{ \begin{array}{l} i) \quad A : H \rightarrow 2^H \text{ is } \alpha\text{-maximal monotone operator } (\alpha \in R), \\ ii) \quad \exists h_0 \in H, \exists r_0, a_1, a_2 > 0 \text{ such that} \\ \quad \quad r_0 \|y\|_{X^*} \leq (y, x - h_0) + a_1 |x|^2 + a_2, \forall [x, y] \in A. \end{array} \right.$$

Theorem 2 *One of the following assumptions implies (H_1) :*

$$\begin{aligned}
& \left\{ \begin{array}{l} \text{a) } A = A_0 + \partial\varphi, \text{ where } A_0 : H \rightarrow H \text{ is a continuous} \\ \text{operator such that:} \\ \quad \exists \alpha \in R : (A_0 x - A_0 y, x - y) + \alpha |x - y|^2 \geq 0 \\ \quad (\text{i.e. } \alpha I + A_0 \text{ is continuous monotone operator on } H) \\ \text{and } \varphi : H \rightarrow]-\infty, +\infty] \text{ is a proper convex} \\ \text{lower-semicontinuous function,} \\ \text{b) } \exists h_0 \in H, R_0 > 0, a_0 > 0 \text{ such that} \\ \quad \varphi(h_0 + x) \leq a_0, \forall x \in X, \|x\|_X \leq R_0, \end{array} \right. \\
& \text{or} \\
& \left\{ \begin{array}{l} \text{a) } \exists V \text{ a separable Banach space such that } V \subset H \subset V^* \\ \text{densely and continuously and } V \cap X \text{ is densely in } X, \\ \text{b) } A : H \rightarrow 2^H \text{ is } \alpha\text{-maximal monotone operator} \\ \text{with } D(A) \subset V \\ \text{c) } \exists a, \beta \in R, a > 0, \text{ such that} \\ \quad (y_1 - y_2, x_1 - x_2) + \beta |x_1 - x_2|^2 \geq a \|x_1 - x_2\|^2, \\ \quad \forall [x_1, y_1], [x_2, y_2] \in A, \\ \text{d) } \exists h_0 \in V, \exists r_0, a_0 > 0 \text{ such that } h_0 + r_0 e \in D(A) \text{ and} \\ \quad \|A^0(h_0 + r_0 e)\|_{V^*} \leq r_0 \text{ for all } e \in V \cap X, \|e\|_X = 1, \end{array} \right. \\
& \text{or} \\
& \left\{ \begin{array}{l} \text{a) } A \text{ is } \alpha\text{-maximal monotone with } \text{int} D(A) \neq \emptyset \\ \text{b) } X = H \end{array} \right.
\end{aligned}$$

Proof. $(H_1\text{-I}) \Rightarrow (H_1)$.

We prove $(H_1\text{-ii})$. Let $r_0 \in [0, R_0]$ such that $|A_0(h_0 + re) - A_0(h_0)| \leq 1$ for all $r \in (0, r_0]$ and $e \in X, \|e\|_X = 1$.

Let $y \in A_0 x + \partial\varphi(x)$. Then $(y - A_0 x, h_0 + r_0 e - x) + \varphi(x) \leq \varphi(h_0 + r_0 e) \leq a_0, \forall e \in X, \|e\|_X = 1$.

Since φ is a convex l.s.c. function, $\exists b_1, b_2 \in R$ such that $\varphi(x) \geq b_1 |x| + b_2$. Hence

$$\begin{aligned}
r_0 (y, e) &\leq (y, x - h_0) + (A_0 x, h_0 + r_0 e - x) - b_1 |x| + a_0 - b_2 \\
&\leq (y, x - h_0) - \alpha |h_0 + r_0 e - x|^2 + \\
&\quad + (A_0(h_0 + r_0 e), h_0 + r_0 e - x) - b_1 |x| + a_0 - b_2
\end{aligned}$$

which clearly yields $(H_1\text{-ii})$.

$(H_1\text{-II}) \Rightarrow (H_1)$

Let $[x, y] \in A$. From $(H_1\text{-II-c})$ we have

$$(A^0(h_0 + r_0 e) - y, h_0 + r_0 e - x) + \beta |h_0 + r_0 e - x|^2 \geq a \|h_0 + r_0 e - x\|_V^2$$

for all $e \in V \cap X, \|e\|_X = 1$ and then

$$\begin{aligned}
r_0(y, e) + a \|h_0 + r_0 e - x\|_V^2 &\leq (y, x - h_0) + \beta |h_0 + r_0 e - x|^2 + (A^0(h_0 + r_0 e), h_0 + r_0 e - x) \\
&\leq (y, x - h_0) + \beta |h_0 + r_0 e - x|^2 + \frac{1}{2a} \|A^0(h_0 + r_0 e)\|_{V^*}^2 \\
&\quad + \frac{a}{2} \|h_0 + r_0 e - x\|_V^2
\end{aligned}$$

which gets (H₁-ii).

(H₁-III) \Rightarrow (H₁).

An α -maximal monotone operator is bounded on $\text{Int}D(A)$. Hence $h_0 \in H$, $r_0 > 0$ exist such that $h_0 + x \in D(A)$, $|A^0(h_0 + x)| \leq r_0$ for all $x \in H$, $|x| \leq r_0$. This operator satisfies (H₁-II) with $V = H = X$, $\beta = 2|\alpha| + 1$, $a = |\alpha| + 1$. So (H₁) holds. \blacksquare

2.2 Generalized deterministic solutions

Let the spaces $X \subset H \subset X^*$ and the equation (2) with the assumptions:

$$\begin{aligned}
i) \quad & A : H \rightarrow 2^H \text{ is } \alpha\text{-maximal monotone operator,} \\
ii) \quad & f \in L^1(0, T; H), \\
iii) \quad & M \in C([0, T]; X), \quad M(0) = 0, \\
iv) \quad & u_0 \in H
\end{aligned} \tag{6}$$

($\alpha \in \mathbb{R}$ given).

Definition 3 A pair of function (u, η) is a generalized (deterministic) solution of equation (2) $((u, \eta) = GD(A; u_0, f, M))$ if the following conditions hold:

$$\begin{aligned}
d_1) \quad & u \in C([0, T]; H), \quad u(t) \in \overline{D(A)} \quad \forall t \in [0, T], \quad u(0) = u_0, \\
d_2) \quad & \eta \in C([0, T]; H) \cap BV([0, T]; X^*), \quad \eta(0) = 0, \\
d_3) \quad & u(t) + \eta(t) = u_0 + \int_0^t f(s)ds + M(t), \quad \forall t \in [0, T], \\
d_4) \quad & \text{there are the sequences } \{u_{0n}\} \subset D(A) \quad \{f_n\} \subset W^{1,1}([0, T]; H), \\
& M_n \in C([0, T]; X) \cap W^{2,1}([0, T]; H) \text{ such that}
\end{aligned}$$

$$\begin{aligned}
i) \quad & u_{0n} \rightarrow u_0 \text{ in } H, \quad f_n \rightarrow f \text{ in } L^1(0, T; H), \\
& M_n \rightarrow M \text{ in } C([0, T]; X), \\
ii) \quad & u_n \rightarrow u \text{ in } C([0, T]; H), \quad \eta_n \rightarrow \eta \text{ in } C([0, T]; H), \\
iii) \quad & \|\eta_n\|_{BV([0, T]; X^*)} \leq C,
\end{aligned} \tag{7}$$

where C is a constant depending only on (A, u_0, f, M, T) and $u_n \in W^{1,\infty}([0, T]; H)$ is the

(strong) solution of the approximating problem

$$\begin{cases} u_n(t) + \int_0^t h_n(s) ds = u_{0n} + \int_0^t f_n(s) ds + M_n(t), \quad t \in [0, T], \\ h_n(t) \in Au_n(t), \quad \text{a.e. } t \in (0, T), \\ \eta_n(t) = \int_0^t h_n(s) ds, \quad t \in [0, T]. \end{cases} \quad (8)$$

We remark that:

$$\begin{aligned} \eta_n &\in W^{2,1}([0, T]; H) \subset C([0, T]; H) \cap BV([0, T]; X^*) \text{ and} \\ (u_n, \eta_n) &= GD(A; u_{0n}, f_n, M_n) \end{aligned}$$

Theorem 4 Assume that

$$\begin{aligned} i) \quad & \text{the operator } A \text{ satisfies } (H_1), \\ ii) \quad & u_0 \in \overline{D(A)}, \\ iii) \quad & f \in L^1(0, T; H), \\ iv) \quad & M \in C([0, T]; X), \quad M(0) = 0. \end{aligned} \quad (9)$$

Then the equation (2) has a unique generalized (deterministic) solution.

Moreover:

$c_1)$ if $(u, \eta) = GD(A; u_0, f, M)$ and $(\bar{u}, \bar{\eta}) = GD(A; \bar{u}_0, \bar{f}, \bar{M})$ are two solutions, then:

$$\begin{aligned} \|u - \bar{u}\|_{C([0, T]; H)}^2 &\leq C[\|u_0 - \bar{u}_0\|^2 + \|f - \bar{f}\|_{L^1(0, T; H)}^2 + \\ &+ \|M - \bar{M}\|_{C([0, T]; H)}^2 + \|M - \bar{M}\|_{C([0, T]; X)} \|\eta - \bar{\eta}\|_{BV([0, T]; X^*)}] \end{aligned} \quad (10)$$

with $C = C(\alpha, T)$ a positive constant, and

$c_2)$ for every equiuniform continuous subset K of $C([0, T]; X)$, $M \in K$, there exists $C_0 = C_0(r_0, h_0, a_1, a_2, T, N_K)^{(*)}$ a positive constant C_0 such that:

$$\|u\|_{C([0, T]; H)}^2 + \|\eta\|_{BV([0, T]; X^*)} \leq C_0[1 + |u_0|^2 + |f|_{L^1(0, T; H)}^2 + \|M\|_{C([0, T]; H)}^2] \quad (11)$$

Proof. Uniqueness. The uniqueness follows from (10), and to prove (10) let $(u_n, \eta_n) = GD(A; u_{0n}, f_n, M_n)$ and $(\bar{u}_n, \bar{\eta}_n) = GD(A; \bar{u}_{0n}, \bar{f}_n, \bar{M}_n)$ where $\{u_{0n}, f_n, M_n\}, \{\bar{u}_{0n}, \bar{f}_n, \bar{M}_n\}$ are chosen as in Definition 3. Then by an easy calculation involving equation (8) we obtain:

$$\begin{aligned} |u_n(t) - M_n(t) - \bar{u}_n(t) + \bar{M}_n(t)|^2 &\leq |u_n(s) - M_n(s) - \bar{u}_n(s) \\ &+ \bar{M}_n(s)|^2 + 2 \int_s^t (M_n - \bar{M}_n, d\eta_n - d\bar{\eta}_n) + 2\alpha \int_s^t |u_n - \bar{u}_n|^2 d\tau \\ &+ 2 \int_s^t (f_n - \bar{f}_n, u_n - M_n - \bar{u}_n + \bar{M}_n) d\tau, \end{aligned}$$

* The constants r_0, h_0, a_1, a_2 are defined in (H_1) ; $N_K \in \mathbb{N}^*$ is a constant of equiuniformly continuity: $\sup\{\|g(t) - g(s)\|_X : |t - s| \leq T/N_K\} \leq r_0/4, \forall g \in K$.

Passing to limit on a subsequences $n_k \rightarrow \infty$, we get:

$$\begin{aligned}
& |u(t) - M(t) - \bar{u}(t) + \bar{M}(t)|^2 \leq |u(s) - M(s) - \bar{u}(s) + \bar{M}(s)|^2 \\
& + 2 \int_s^t (M(\tau) - \bar{M}(\tau), d\eta(\tau) - d\bar{\eta}(\tau)) + 2\alpha \int_s^t |u - \bar{u}|^2 d\tau \\
& + 2 \int_s^t (f - \bar{f}, u - M - \bar{u} + \bar{M}) d\tau
\end{aligned} \tag{12}$$

for all $0 \leq s \leq t \leq T$, which implies clearly (10).

Existence. If Y is a Banach space and $g : [0, T] \rightarrow Y$ is a continuous function we set

$$m_Y(\delta; g) = \sup \{ \|g(t) - g(s)\|_Y \mid t, s \in [0, T], |t - s| \leq \delta \}$$

(modulus of continuity).

For u_0, f, M given as in (9) these exist the sequences

$$\begin{aligned}
& \{u_{0n}\} \subset D(A), \quad \{f_n\} \subset W^{1,1}([0, T]; H), \\
& \{M_n\} \subset C([0, T]; X) \cap W^{2,1}([0, T]; H), \quad M_n(0) = 0
\end{aligned}$$

such that $m_X(\delta, M_n) \leq m_X(\delta, M)$, $\forall n \in N^*$, $\forall \delta > 0$,

$$u_{0n} \rightarrow u_0 \text{ in } H,$$

$$f_n \rightarrow f \text{ in } L^1(0, T; H),$$

$$M_n \rightarrow M \text{ in } C([0, T]; X).$$

The conditions on M_n are satisfied setting, for example,

$$M_n(t) = n \int_R \rho(n(t-s) - 1) \widetilde{M}(s) ds = \int_R \rho(r) \widetilde{M}\left(t - \frac{1+r}{n}\right) dr,$$

where $\rho \in C_0^\infty(R)$, $\rho(-r) = \rho(r) \geq 0 \quad \forall r \in R$, $\rho(r) = 0 \quad \forall |r| \geq 1$, $\int_R \rho(r) dr = 1$ and

$$\widetilde{M}(t) = \begin{cases} M(0), & t < 0 \\ M(t), & t \in [0, T] \\ M(T), & t > T. \end{cases}$$

Let K be an equiuniformly continuous subset of $C([0, T]; X)$ which contains M , and let $\delta = T/N_0$ sufficiently small such that

$$m_X\left(\frac{T}{N_0}, M\right) \leq \frac{r_0}{4}$$

The approximating problem:

$$\begin{aligned}
& u_n'(t) + Au_n(t) \ni f_n(t) + M_n'(t) \\
& u_n(0) = u_{0n}
\end{aligned} \tag{13}$$

has a unique strong solution $u_n \in W^{1,\infty}([0, T]; H)$ and the sequence η_n defined by $\eta'_n(t) = f_n(t) + M'_n(t) - u'_n(t) \in Au_n(t)$, $\eta_n(0) = 0$, is in $W^{2,1}([0, T]; H)$. We multiply equation (13) by $u_n - M_n - h_0$ and integrate on $[0, t]$; the equality

$$\begin{aligned} & |u_n(t) - M_n(t) - h_0|^2 + 2 \int_0^t \left(\eta'_n(s), u_n(s) - h_0 \right) ds = |u_{0n} - h_0|^2 \\ & + 2 \int_0^t (f_n(s), u_n(s) - M_n(s) - h_0) ds + 2 \int_0^t \left(\eta'_n(s), M_n(s) \right) ds \end{aligned} \quad (14)$$

follows. Let $0 = r_0 < r_1 < \dots < r_m = T$, $r_{i+1} - r_i = \frac{T}{N_0}$, $i = \overline{0, m-1}$ and $t \in [r_k, r_{k+1}]$. Denote $t_i = r_i$ if $i \in \overline{0, k}$, $t_{k+1} = t$. Then

$$\begin{aligned} & \int_0^t \left(\eta'_n(s), M_n(s) \right) ds = \\ & = \sum_{i=0}^k \left[\int_{t_i}^{t_{i+1}} (M_n(s) - M_n(t_i), d\eta_n(s)) + (M_n(t_i), \eta_n(t_{i+1}) - \eta_n(t_i)) \right] \\ & \leq m_X \left(\frac{T}{N_0}, M_n \right) \|\eta_n\|_{BV([0, T]; X^*)} + 2N_0 \|M_n\|_{C([0, T]; H)} \|\eta_n\|_{C([0, T]; H)} \\ & \leq \frac{r_0}{4} \|\eta_n\|_{BV([0, T]; X^*)} + 2N_0 \|M_n\|_{C([0, T]; H)} \left[\int_0^T |f_n(s)| ds + \right. \\ & \quad \left. + \sup_{s \in [0, t]} |u_n(s) - M_n(s) - h_0| + |h_0| \right] \end{aligned}$$

Hence by (H_1)

$$\begin{aligned} & \sup_{s \in [0, t]} |u_n(s) - M_n(s) - h_0|^2 + 2r_0 \int_0^t \|\eta'_n(s)\|_{X^*} ds \leq 2|u_{0n} - h_0|^2 \\ & + 18 \left(\int_0^T |f_n(s)| ds \right)^2 + 4|a_2|T + \frac{1}{2} \sup_{s \in [0, t]} |u_n(s) - M_n(s) - h_0|^2 \\ & + 4|a_1| \int_0^t \sup_{\tau \in [0, s]} |u_n(\tau)|^2 d\tau + r_0 \|\eta_n\|_{BV([0, T]; X^*)} + 18N_0^2 \|M_n\|_{C([0, T]; H)}^2 \end{aligned}$$

We obtain

$$\begin{aligned} & \|u_n\|_{C([0, T]; H)}^2 + \|\eta_n\|_{BV([0, T]; X^*)} \\ & \leq C \left[1 + |u_{0n}|^2 + \|f_n\|_{L^1(0, T; H)}^2 + \|M_n\|_{C([0, T]; H)}^2 \right] \leq C_1, \end{aligned} \quad (15)$$

where $C = C(r_0, h_0, N_0, T, a_1, a_2)$ and $C_1 = C_1(A, T, u_0, f, M)$ are positive constants.

Since $(u_n, \eta_n) = GD(A; u_{0n}, f_n, M_n)$, $n \in \mathbb{N}$, then by (10) which we already proved we have

$$\begin{aligned} & \|u_n - u_m\|_{C([0, T]; H)}^2 \leq C[|u_{0n} - u_{0m}|^2 + \|f_n - f_m\|_{L^1(0, T; H)}^2 \\ & \quad + \|M_n - M_m\|_{C([0, T]; H)}^2 + \|M_n - M_m\|_{C([0, T]; X)} \|\eta_n - \eta_m\|_{BV([0, T]; X^*)}] \\ & \leq C_2[|u_{0n} - u_{0m}|^2 + \|f_n - f_m\|_{L^1(0, T; H)}^2 + \|M_n - M_m\|_{C([0, T]; H)}^2 + \\ & \quad + \|M_n - M_m\|_{C([0, T]; X)}] \end{aligned}$$

where $C_2 = C_2(A, T, u_0, f, M) > 0$. Hence there exists $u \in C([0, T]; H)$ such that

$$\begin{aligned} u_n &\rightarrow u \quad \text{in } C([0, T]; H) \\ \eta_n &= u_{0n} + \int_0^\cdot f_n(s) ds + M_n - u_n \rightarrow u_0 + \int_0^\cdot f ds + M - u \\ &\quad \text{in } C([0, T]; H) \\ \eta_n &\rightarrow \eta \quad \text{weak star in } BV([0, T]; X^*) \end{aligned}$$

an by (15) the inequality (11) follows. The proof is complete. \blacksquare

From (14) and (H₁) as $n \rightarrow \infty$, we have:

Remark 5 *If $(u, \eta) = GD(A; u_0, f, M)$, then:*

$$\begin{aligned} |u(t) - M(t) - h_0|^2 + 2r_0 \|\eta\|_{BV([0, t]; X^*)} &\leq |u_0 - h_0|^2 + 2|a_2|t \\ &+ 2|a_1| \int_0^t |u(s)|^2 ds + 2 \int_0^t (f(s), u(s) - M(s) - h_0) ds + 2 \int_0^t (M(s), d\eta(s)), \end{aligned}$$

for all $t \in [0, T]$.

In the next section this inequality will be used for some estimates of the stochastic generalized solutions.

Corollary 6 *Let the assumptions of Theorem 4 be satisfied A_ε^α the Yosida approximation of the operator $A + \alpha I$ and u_ε , with $0 < \varepsilon < \frac{1}{|\alpha|+1}$, the solution of the penalized equation*

$$\begin{aligned} du_\varepsilon(t) + (A_\varepsilon^\alpha u_\varepsilon(t) - \alpha u_\varepsilon(t)) dt &= f(t) dt + dM(t), \\ u_\varepsilon(0) &= u_0, \\ \eta_\varepsilon(t) &= \int_0^t (A_\varepsilon^\alpha u_\varepsilon(s) - \alpha u_\varepsilon(s)) ds. \end{aligned} \tag{16}$$

Then as $\varepsilon \rightarrow 0$:

$$\begin{aligned} u_\varepsilon &\rightarrow u \quad \text{in } C([0, T]; H), \\ \eta_\varepsilon &\rightarrow \eta \quad \text{in } C([0, T]; H), \\ \eta_\varepsilon &\xrightarrow{w^*} \eta \quad (\text{weak star}) \text{ in } BV([0, T]; X^*). \end{aligned}$$

Remark 7 *We remark that $u_\varepsilon = v_\varepsilon + M$, where $v_\varepsilon \in C^1([0, T]; H)$ is the strong solution of the equation:*

$$\begin{cases} v'_\varepsilon + A_\varepsilon^\alpha(v_\varepsilon + M(t)) - \alpha(v_\varepsilon + M(t)) = f(t) \\ v_\varepsilon(0) = u_0 \end{cases}$$

Proof of Corollary 6. Let u_{0n}, f_n, M_n as in the proof of Theorem 4 and $(u_\varepsilon^n, \eta_\varepsilon^n)$ the strong solution of the equation (16) corresponding to (u_{0n}, f_n, M_n) . Then by Proposition 3 $u_\varepsilon^n \rightarrow u^n$ in $C([0, T]; H)$ as $\varepsilon \rightarrow 0$.

Also since $A_\varepsilon^\alpha x - \alpha J_\varepsilon^\alpha x \in A(J_\varepsilon^\alpha x)$, $J_\varepsilon^\alpha x = x - \varepsilon A_\varepsilon^\alpha x$ and $|J_\varepsilon^\alpha x| \leq |x| + (1 + |\varepsilon - 1|) |J_1^\alpha 0|$ then by (H₁-ii) we have $r_0 \|A_\varepsilon^\alpha x - \alpha x\|_{X^*} \leq (A_\varepsilon^\alpha x - \alpha x, x - h_0) + b_1 |x|^2 + b_2$, where $b_i = b_i(\alpha, h_0, a_1, a_2) > 0$, $i = 1, 2$.

Since $(u_\varepsilon, \eta_\varepsilon) = GD(A_\varepsilon^\alpha - \alpha I; u_0, f, M)$ and $(u_\varepsilon^n, \eta_\varepsilon^n) = GD(A_\varepsilon^\alpha - \alpha I; u_{0n}, f_n, M_n)$, then by (10) and (11)

$$\begin{aligned} \|u_\varepsilon\|_{C([0,T];H)}^2 + \|\eta_\varepsilon\|_{BV([0,T];X^*)} &\leq C_3[1 + |u_0|^2 + |f|_{L^1(0,T;H)}^2 + \|M\|_{C([0,T];H)}^2], \\ \|u_\varepsilon^n\|_{C([0,T];H)}^2 + \|\eta_\varepsilon^n\|_{BV([0,T];X^*)} &\leq C_3[1 + |u_{0n}|^2 + |f_n|_{L^1(0,T;H)}^2 + \|M_n\|_{C([0,T];H)}^2] \end{aligned}$$

and

$$\begin{aligned} |u_\varepsilon - u_\varepsilon^n|_{C([0,T];H)}^2 &\leq C_4[|u_0 - u_{0n}|^2 + \|f - f_n\|_{L^1(0,T;H)}^2 \\ &\quad + \|M - M_n\|_{C([0,T];H)}^2 + \|M - M_n\|_{C([0,T];X)}] = \alpha_n \end{aligned}$$

where C_3, C_4 are constants independent of ε and n , and $\lim_{n \rightarrow \infty} \alpha_n = 0$. Also from the proof of Theorem 4 we have

$$\|u - u_n\|_{C([0,T];H)}^2 \leq \alpha_n$$

Now from

$$\|u_\varepsilon - u\|_{C([0,T];H)}^2 \leq 3[\|u_\varepsilon - u_\varepsilon^n\|_{C([0,T];H)}^2 + \|u_\varepsilon^n - u_n\|_{C([0,T];H)}^2 + \|u_n - u\|_{C([0,T];H)}^2]$$

we have

$$\limsup_{\varepsilon \rightarrow 0} \|u_\varepsilon - u\|_{C([0,T];H)} \leq 6\alpha_n \quad \text{for all } n \in \mathbb{N}^*$$

and the conclusions of Corollary 6 follows easily. ■

Corollary 8 *If we substitute the assumption (9-i) of Theorem 4 by*

$$\text{operator } A \text{ satisfies } (H_1\text{-III}), \tag{17}$$

then $(u, n) = GD(A; u_0, f, M)$ if and only if (u, η) is the solution of the following problem:

$$\begin{aligned} \text{i)} \quad & u \in C([0, T]; H), \quad u(t) \in \overline{D(A)} \quad \forall t \in [0, T], \quad u(0) = u_0, \\ \text{ii)} \quad & \eta \in C([0, T]; H) \cap BV([0, T]; H), \quad \eta(0) = 0, \\ \text{iii)} \quad & u(t) + \eta(t) = u_0 + \int_0^t f(s) ds + M(t), \quad \forall t \in [0, T], \\ \text{iv)} \quad & \int_s^t (u(\tau) - x, d\eta(\tau) - y d\tau) + \alpha \int_s^t |u(\tau) - x|^2 d\tau \geq 0 \\ & \forall 0 \leq s \leq t \leq T, \text{ and } \forall [x, y] \in A \end{aligned} \tag{18}$$

Proof. If $(u, \eta) = GD(A; u_0, f, M)$ then (u, η) satisfies (18-i,ii,iii). Also, since $\eta'_n + \alpha u_n \in (A + \alpha I)(u_n)$ and $y + \alpha x \in (A + \alpha I)(x)$ for $[x, y] \in A$, then by monotony of $A + \alpha I$ the inequality (18-iv) follows as $n \rightarrow \infty$.

To finish the proof of Corollary 5 we have to prove only the uniqueness of the solution of the problem (18). It is clearly that (18-iv) gives

$$\int_s^t (u(\tau) - a(\tau), d\eta(\tau) - b(\tau) d\tau) + \alpha \int_s^t |u(\tau) - a(\tau)|^2 d\tau \geq 0 \tag{19}$$

for all $a(\tau), b(\tau)$ step functions on $[0, T]$ such that $[a(\tau), b(\tau)] \in A, \forall \tau \in [0, T]$, and then for all $a, b \in C([0, T]; H)$, $[a(\tau), b(\tau)] \in A \forall \tau \in [0, T]$. Now the uniqueness follows by a standard argument. We write (19) for $u = u_1$ and $u = u_2$, $a = J_\varepsilon^\alpha\left(\frac{u_1+u_2}{2}\right)$, $b = A_\varepsilon^\alpha\left(\frac{u_1+u_2}{2}\right) - \alpha J_\varepsilon^\alpha\left(\frac{u_1+u_2}{2}\right)$; by addition of the two inequalities and by passing to limit for $\varepsilon \rightarrow 0$ the following inequality follows

$$\frac{1}{2} \int_s^t (u_1(\tau) - u_2(\tau), d\eta_1(\tau) - d\eta_2(\tau)) + \alpha \int_s^t |u_1(\tau) - u_2(\tau)|^2 d\tau \geq 0, \quad (20)$$

for all $0 \leq s \leq t \leq T$.

From (20-iii) we have $u_1(t) - u_2(t) = \eta_2(t) - \eta_1(t)$. Hence by (20):

$$|u_1(t) - u_2(t)|^2 \leq 4\alpha \int_0^t |u_1(\tau) - u_2(\tau)|^2 d\tau \quad \forall t \in [0, T],$$

which implies $u_1 = u_2$. ■

Corollary 9 *If we substitute the assumption (9-i) of Theorem 4 by*

$$\text{operator } A \text{ satisfies } (H_1 - II), \quad (21)$$

then moreover the generalized solution $u \in L^2(0, T; V)$.

Proof. By the assumption (H₁-II-c) we have that u_n is a Cauchy sequence in $L^2(0, T; V)$. ■

2.3 An example

Let D be an open bounded subset of \mathbb{R}^d with a sufficiently smooth boundary Γ , and let $\beta \subset \mathbb{R} \times \mathbb{R}$ be a maximal monotone graph or equivalent $\beta = \partial j$, where $j : \mathbb{R} \rightarrow]-\infty, +\infty]$ is a convex lower-semicontinuous function. We assume that $\exists b_0 \in \mathbb{R}, \exists \varepsilon_0 > 0$ such that $[b_0 - \varepsilon_0, b_0 + \varepsilon_0] \subset \text{Dom}(j)$. Also let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that

$$\begin{aligned} (g(r) - g(q))(r - q) + \alpha |r - q|^2 &\geq 0 \quad \forall r, q \in \mathbb{R} \\ |g(r)| &\leq b(1 + |r|) \end{aligned}$$

(α, b are some given positive constants). Consider the following problem

$$\begin{cases} du(t) - \Delta u(t) dt + g(u(t)) dt = f(t) dt + dM(t) \text{ on }]0, T[\times D, \\ -\frac{\partial u(t, x)}{\partial n} \in \beta(u(t, x)) \text{ on }]0, T[\times \Gamma, \\ u(0, x) = u_0(x), \text{ on } D \end{cases} \quad (22)$$

or equivalent

$$\begin{cases} du(t) + \partial \varphi(u(t))(dt) + g(u(t)) dt \ni f(t) dt + dM(t) \\ u(0) = u_0 \end{cases}$$

where $\varphi : H = L^2(D) \rightarrow]-\infty, +\infty]$ is the convex l.s.c. function given by

$$\varphi(u) = \begin{cases} \frac{1}{2} \int_D |\text{grad } u|^2 dx + \int_\Gamma j(u) d\sigma, & \text{if } u \in H^1(D), j(u) \in L^1(\Gamma) \\ +\infty, & \text{otherwise} \end{cases}$$

The assumptions from (H₁-I) are satisfied for $(A_0 u)(x) = g(u(x))$ and the Sobolev space $H^k(D) = X$, $k > \frac{d}{2}$. Hence if $f \in L^1(0, T; L^2(D))$, $u_0 \in H^1(D)$, $j(u_0) \in L^1(\Gamma)$, $M \in C([0, T]; H^k(D))$ then the equation (22) has a unique generalized solution $u \in C([0, T]; L^2(D))$ in the sense of Definition 3.

3 Stochastic evolution equations

3.1 Preliminaries

A. We assume as given a filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ which satisfy the usual hypotheses i.e. $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and $\{\mathcal{F}_t, t \geq 0\}$ is a increasing right continuous sub- σ -algebras of \mathcal{F} . We shall say that $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ is a stochastic base.

If Y is a real separable Banach space we denote by $L_{ad}^r(\Omega; C([0, T]; Y))$, $r \geq 0$, the closed linear subspace (Banach space for $r \in [1, \infty)$ and metric space for $0 \leq r < 1$; the metric of convergence in probability for $r = 0$) of adapted stochastic processes $f \in L^r(\Omega, \mathcal{F}, \mathbb{P}; C([0, T]; Y))$. Similarly $L_{ad}^r(\Omega; L^q(0, T; X))$, $r \geq 0$, $q \in [1, \infty)$, denoted the Banach space for $r \geq 1$ or the metric space for $0 \leq r < 1$ of measurable stochastic processes $f \in L^r(\Omega; L^q(0, T; Y))$ such that $f(\cdot, t)$ is \mathcal{F}_t -measurable a.e. $t \in (0, T)$.

B. If H is a real separable Hilbert space we denote by $\mathcal{M}^p(0, T; H)$, $p \in [1, \infty)$, the space of continuous p -martingales M , that is

$$\begin{aligned} (i) \quad & M \in L_{ad}^0(\Omega; C([0, T]; H)), \\ (ii) \quad & \mathbb{E} |M_t|^p < \infty, \quad \text{for all } t \geq 0, \\ (iii) \quad & M(\omega, 0) = 0, \quad a.s. \omega \in \Omega, \\ (iv) \quad & \mathbb{E}(M(t) | \mathcal{F}_s) = M(s), \quad a.s., \text{ if } 0 \leq s \leq t \leq T. \end{aligned} \tag{23}$$

If $M \in \mathcal{M}^2(0, T; H)$ then by Doob-Meyer decomposition, there exists a unique stochastic process $\langle M \rangle \in L_{ad}^1(\Omega; C([0, T]; R))$ such that

$$\begin{aligned} i) \quad & t \longmapsto \langle M \rangle(\omega, t) \text{ is increasing } \mathbb{P}\text{-a.s.}, \\ ii) \quad & |M|_H^2 - \langle M \rangle \in \mathcal{M}^1(0, T; R). \end{aligned} \tag{24}$$

Moreover

$$\begin{aligned} a) \quad & \mathbb{E} \sup_{s \in [0, T]} |M(s)| \leq 3 \mathbb{E} \sqrt{\langle M \rangle(T)}, \\ b) \quad & \mathcal{M}^p(0, T; H) \text{ is a closed linear subspace of} \\ & L_{ad}^p(\Omega; C([0, T]; H)), \text{ for all } p \in (1, \infty). \end{aligned} \tag{25}$$

On $\mathcal{M}^p(0, T; H)$, $p > 1$, it is defined the norm $\|M\|_{\mathcal{M}^p} = (\mathbb{E} |M(T)|_H^p)^{1/p}$; in the case $p > 1$ this norm is equivalent on $\mathcal{M}^p(0, T; H)$ to usual norm from $L^p(\Omega; C([0, T]; H))$. The space $(\mathcal{M}^2(0, T; H), \|\cdot\|_{\mathcal{M}^2})$ is a Hilbert space.

For any $f \in L_{ad}^r(\Omega; C([0, T]; H))$, $0 \leq r < \infty$ and $M \in \mathcal{M}^2(0, T; H)$ the stochastic integral $I(f; M)(t) = \int_0^t (f(s), dM(s))$ is well defined and has the properties:

- a) $I : L_{ad}^2(\Omega; C([0, T]; H)) \times \mathcal{M}^2(0, T; H) \rightarrow \mathcal{M}^1(0, T; \mathbb{R})$
is continuous, and also
 $I : L_{ad}^0(\Omega; C([0, T]; H)) \times \mathcal{M}^2(0, T; H) \rightarrow L_{ad}^0(\Omega; C([0, T]; \mathbb{R}))$
is continuous;
- b) $|M(t)|^2 = 2 \int_0^t (M(t), dM(t)) + \langle M \rangle(t)$, $\forall t \in [0, T]$, a.s. $\omega \in \Omega$;
- c) if $f \in L_{ad}^2(\Omega; C([0, T]; H))$ then
$$\mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t (f(s), dM(s)) \right| \leq 3 \mathbb{E} \left[\sup_{t \in [0, T]} |f(t)|_H \sqrt{\langle M \rangle(T)} \right] \quad (26)$$
- d) if $0 = t_0 < t_1 < \dots < t_n = T$, $\delta_n = \max_{i=0, n-1} |t_{i+1} - t_i|$ and

$$I_n(f)(t) = \sum_{i=0}^{n-1} (f(t_i), M(t_{i+1} \wedge t) - M(t_i \wedge t)) \quad \text{then}$$

$$I(f; M) = \lim_{\delta_n \rightarrow 0} I_n(f) \begin{cases} \text{in } \mathcal{M}^1(0, T; \mathbb{R}), \text{ for} \\ f \in L_{ad}^2(\Omega; C([0, T]; H)) \\ \text{in } L_{ad}^0(\Omega; C([0, T]; \mathbb{R})), \text{ for} \\ f \in L_{ad}^0(\Omega; C([0, T]; H)) \end{cases}$$

C. Let $(U | \cdot |_U)$ be a real separable Hilbert space and $Q : U \rightarrow U$ be a linear operator. We shall assume

- i) $Qu = \sum_i \lambda_i (u, e_i)_U e_i$, where
 $\lambda_i \geq 0$, $\sum_i \lambda_i < \infty$, $\{e_i\}$ orthonormal basis in U ,
- ii) $W \in \mathcal{M}^2(0, T; U)$ is a U -valued Wiener process with covariance operator Q .

Hence

- a) $W(\omega, t) = \sum_i \sqrt{\lambda_i} \beta_i(\omega, t) e_i$,
- b) $\beta_i \in L_{ad}^2(\Omega; C([0, T]; \mathbb{R}))$ are independent real Brownian motions with respect to the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$
- c) $W \in \mathcal{M}^p(0, T; U)$, $\forall p \geq 1$.

To define the stochastic integral with respect to a Q -Wiener process one introduces the Hilbert space $U_0 = Q^{1/2}(U)$ endowed with the inner product

$$(u, v)_{U_0} = \sum_{\lambda_k \neq 0} \frac{1}{\lambda_k} (u, e_k)_U (v, e_k)_U$$

and the space of all Hilbert-Schmidt operators $L_0^2 = L^2(U_0, H)$. The space L_0^2 is a separable Hilbert space, equipped with the Hilbert-Schmidt norm:

$$|f|_Q^2 \triangleq \|f\|_{\mathcal{L}_0^2}^2 = \sum_i \lambda_i |f e_i|^2 = \sum_i \left| f Q^{1/2} e_i \right|^2 = \|f Q^{1/2}\|_{HS}^2 = \text{tr } f Q f^*.$$

The stochastic integral $I(f)(\omega, t) = \int_0^t f(\omega, s) dW(\omega, s)$ is well defined for $f \in L_{ad}^r(\Omega; L^2(0, T; \mathcal{L}_0^2))$, $0 \leq r < \infty$ and

- a) $I(f) \in L_{ad}^r(\Omega; C([0, T]; H))$,
 - b) $\mathbb{E} I(f)(t) = 0$, for $r \geq 1$,
 - c) $\mathbb{E} |I(f)(t)|^2 = \mathbb{E} \int_0^t |f(s)|_Q^2 ds$, for $r \geq 2$,
 - d) (Burkholder-Davis-Gundy inequality): $\forall r > 0, \exists c_r, C_r > 0$ ^(†) : (29)
- $$c_r \mathbb{E} \left(\int_0^T |f|_Q^2 ds \right)^r / 2 \leq \mathbb{E} \sup_t \in [0, T] |I(f)(t)|^r \leq C_r \mathbb{E} \left(\int_0^T |f|_Q^2 ds \right)^r / 2,$$
- e) $I(f) \in \mathcal{M}^r(0, T; H)$, for $r \in [1, \infty)$.

D. Let $X \subset H \cong H^* \subset X^*$ be the spaces defined as in Section 2 (Subsection 3: Preliminaries).

Proposition 10 (*Integration by parts*)

- a) if $m \in L^0(\Omega; C([0, T]; X)) \cap \mathcal{M}^2(0, T; H)$
 $\eta \in L_{ad}^0(\Omega; C([0, T]; H)) \cap L^0(\Omega; BV(0, T; X^*))$, then

$$\int_0^t (m(s), d\eta(s)) = (m(t), \eta(t)) - \int_0^t (\eta(s), dm(s))$$

for all $t \in [0, T]$, a.s. $\omega \in \Omega$.
- b) Moreover if

$$u(t) + \eta(t) = u_0 + \int_0^t f(s) ds + m(t), \forall t \in [0, T], \text{ a.s.,}$$

where $u \in L_{ad}^0(\Omega; C([0, T]; H))$,
 $u_0 \in L^0(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, $f \in L_{ad}^0(\Omega; L^1(0, T; H))$

[†] $C_2 = 4$ (Doob inequality), $C_r \leq 3$ if $0 < r \leq 1$ and $C_r \leq 9(2r)^r$ if $r > 1$.

then

$$\begin{aligned}
|u(t) - m(t)|^2 &= |u(t)|^2 - 2 \int_0^t (u(s), dm(s)) - 2 \int_0^t (m(s), f(s)) ds + \\
&+ 2 \int_0^t (m(s), d\eta(s)) - \langle m \rangle(t) \quad \text{for all } t \in [0, T], \text{ a.s. } \omega \in \Omega.
\end{aligned} \tag{30}$$

Proof. Denote $F(t) = \int_0^t f(s) ds$. Let $0 = t_0 < t_1 < \dots < t_n = t$, $\frac{t}{n} = t_{i+1} - t_i$ and $g(t_i) = g_i$. From the definition of Riemann-Stieltjes and the properties of stochastic integral (26) we have for \mathbb{P} -a.s. $\omega \in \Omega$ (on a subsequence n_k denoted also n)

$$\begin{aligned}
\int_0^t (m(s), d\eta(s)) &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} (m_{i+1}, \eta_{i+1} - \eta_i) \\
&= \lim_{n \rightarrow \infty} [(m_n, \eta_n) - (m_0, \eta_0) - \sum_{i=0}^{n-1} (\eta_i, m_{i+1} - m_i)] \\
&= (m(t), \eta(t)) - \int_0^t (\eta(s), dm(s)) \\
&= (m(t), u_0 + F(t) + m(t) - u(t)) \\
&\quad - \int_0^t (u_0 + F(s) + m(s) - u(s), dm(s)) \\
&= [(m(t), F(t)) - \int_0^t (F(s), dm(s))] \\
&\quad + [\int_0^t (u(s), dm(s)) - (m(t), u(t))] \\
&\quad + [|m(t)|^2 - \int_0^t (m(s), dm(s))] \\
&= \int_0^t (f(s), m(s)) ds + [\int_0^t (u(s), dm(s)) - (m(t), u(t))] \\
&\quad + [\frac{1}{2} |m(t)|^2 + \frac{1}{2} \langle m \rangle(t)] \\
&= \int_0^t (f(s), m(s)) ds + \int_0^t (u(s), dm(s)) \\
&\quad + \frac{1}{2} |u(t) - m(t)|^2 - \frac{1}{2} |u(t)|^2 + \frac{1}{2} \langle m \rangle(t).
\end{aligned}$$

■

E. Finally it is interesting to recall a general Pardoux's result from [11]. We shall see that our results means a generalization, also, for unbounded operator but in the multivalued case.

Let $(H, (\cdot, \cdot), |\cdot|)$ be a real separable Hilbert space and let $(V, \|\cdot\|)$ be a real separable reflexive Banach space with the dual $(V^*, \|\cdot\|_{V^*})$ strictly convex space. Assume $V \subset H \cong H^* \subset V^*$ with continuous densely embeddings.

Consider the equation

$$\begin{aligned} dy(t) + A(t, y(t)) dt &= f(t) dt + B(t, y(t)) dW(t) + dM(t) \\ y(0) &= y_0, \quad t \in [0, T], \end{aligned} \quad (31)$$

where $A(t, \cdot) : V \rightarrow V^*$ defined for a.e. $t \in (0, T)$ satisfies: there exist the constants $p > 1$, $a, \alpha > 0$, $d, d_0 \in \mathbb{R}$ such that for all $u, v, z \in V$ the following properties hold a.e. $t \in (0, T)$:

$$\begin{aligned} i) \quad & 2 \langle A(t, u), u \rangle + d|u|^2 + d_0 \geq \alpha \|u\|^p, \\ ii) \quad & 2 \langle A(t, u) - A(t, v), u - v \rangle + d|u - v|^2 \geq 0, \\ iii) \quad & \|A(t, u)\|_*^{p'} \leq a(1 + \|u\|^p), \quad 1/p + 1/p' = 1, \\ iv) \quad & s \mapsto \langle A(t, u + sv), z \rangle : \mathbb{R} \rightarrow \mathbb{R} \text{ is a continuous function,} \\ v) \quad & s \mapsto \langle A(s, u), v \rangle :]0, T[\rightarrow \mathbb{R} \text{ is Lebesgue measurable,} \end{aligned} \quad (32)$$

and $B(t, \cdot) : H \rightarrow L^2(U_0, H)$ defined for a.e. $t \in]0, T[$ satisfies: $\exists L, b > 0$ such that for all $u, v \in H$, $w \in U_0$ we have a.e. $t \in]0, T[$:

$$\begin{aligned} i) \quad & |B(t, u) - B(t, v)|_Q^2 \leq L|u - v|^2, \\ ii) \quad & |B(t, u)|_Q^2 \leq b(1 + |u|^2), \\ iii) \quad & s \mapsto (B(s, u)w, v) :]0, T[\rightarrow \mathbb{R} \text{ is Lebesgue measurable} \end{aligned} \quad (33)$$

Also it is assumed:

$$\begin{aligned} i) \quad & y_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H), \\ ii) \quad & f = f_1 + f_2, \quad f_1 \in L_{ad}^2(\Omega; L^1(0, T; H)), \quad f_2 \in L_{ad}^{p'}(\Omega \times]0, T[; V^*), \\ iii) \quad & M \in \mathcal{M}^2(0, T; H), \\ iv) \quad & \{W(t), t \geq 0\} \text{ is a } Q\text{-Wiener process.} \end{aligned} \quad (34)$$

Theorem 11 (*E. Pardoux [11]*). *Under the assumptions (32) (33) and (34) the equation (31) has a unique solution $y \in L_{ad}^p(\Omega \times]0, T[; V) \cap L_{ad}^2(\Omega; C([0, T]; H))$. The solution y satisfies*

a) *(Energy Equality)*

$$\begin{aligned} |y(t)|^2 + 2 \int_0^t \langle A(s, y(s)), y(s) \rangle ds &= |y_0|^2 + 2 \int_0^t \langle f(s), y(s) \rangle ds + 2 \int_0^t (y(s), B(s, y(s)) dW(s)) \\ &+ 2 \int_0^t (y(s), dM(s)) + \langle M - \int_0^\cdot B(s, y(s)) dW(s) \rangle(t) \quad \text{for all } t \in [0, T], \quad \text{a.e. } \omega \in \Omega. \end{aligned}$$

b) $m_1(\cdot) = \int_0^\cdot (y(s), B(s, y(s)) dW(s))$, $m_2(\cdot) = \int_0^\cdot (y(s), dM(s))$ are martingales from $M^1(0, T; R)$.

3.2 α -Monotone SDE with additive noise

Let $X \subset H \cong H^* \subset X^*$ be the spaces defined at the beginning of Section 2 that is: $(H, |\cdot|)$ is a real separable Hilbert space and $(X, \|\cdot\|)$ is a real separable Banach space with the dual $(X^*, \|\cdot\|_*)$ separable too. The inclusion mapping $X \subset H$ is continuous and X is dense in H .

We shall assume given a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$. Consider the multivalued stochastic differential equation:

$$\begin{cases} du(t) + Au(t) dt \ni f(t) dt + dM(t), \\ u(0) = u_0, \quad t \in [0, T], \end{cases} \quad (35)$$

where we put the assumptions:

$$\begin{aligned} (A_1) \quad & \begin{cases} A : H \rightarrow 2^H \text{ satisfies } (H_1), \text{ that is} \\ i) \ A : H \rightarrow 2^H \text{ is } \alpha\text{-maximal monotone operator } (\alpha \in \mathbb{R}), \\ ii) \ \exists h_0 \in H, \exists r_0, a_1, a_2 > 0 \text{ such that} \\ r_0 \|y\|_{X^*} \leq (y, x - h_0) + a_1 |x|^2 + a_2, \forall [x, y] \in A. \quad (\dagger) \end{cases} \\ \text{and} \\ (A_2) \quad & \begin{cases} i) \ u_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H), \ u_0(\omega) \in \overline{D(A)}, \ \mathbb{P}\text{-a.s. } \omega \in \Omega, \\ ii) \ f \in L^2_{ad}(\Omega; L^1(0, T; H)), \\ iii) \ M \in \mathcal{M}^2(0, T; H). \end{cases} \end{aligned}$$

We mention that by Theorem 4 if $M \in L^0(\Omega; C([0, T]; X))$ then equation (35) has a unique (determinist) solution

$$u(\omega, \cdot) = GD(A; u_0(\omega), f(\omega, \cdot), M(\omega, \cdot)) \text{ a.s. } \omega \in \Omega.$$

In the sequel using the martingale properties we shall extend our determinist results to the case when M has not X -valued continuous trajectories. The corresponding solution will be denoted by $GS(A; u_0, f, M)$. Certainly we shall use the determinist result approximating $M \in \mathcal{M}^2(0, T; H)$ by

$$\begin{aligned} \overline{M}_n & \in L^2_{ad}(\Omega; C([0, T]; X)) \cap \mathcal{M}^2(0, T; H) \\ \overline{M}_n & \rightarrow M \text{ in } \mathcal{M}^2(0, T; H). \end{aligned} \quad (36)$$

Remark that putting

$$\overline{M}_n(t) = \sum_{i=1}^n (M(t), h_i) h_i, \quad (37)$$

where $\{h_i, i \in \mathbb{N}^*\} \subset X$ is an orthonormal basis in H , then \overline{M}_n satisfies (36).

[†]see also Theorem 2.2

Definition 12 A pair of stochastic processes (u, η) is the generalized stochastic solution of the stochastic evolution equation (35), $(u, \eta) = GS(A; u_0, f, M)$, if

- $s_1)$ $u \in L_{ad}^0(\Omega; C([0, T]; H)); u(0) = u_0$ a.s. and
 $u(t) \in \overline{D(A)}, \forall t \in [0, T]$ a.s.,
- $s_2)$ $\eta \in L_{ad}^0(\Omega; C([0, T]; H)) \cap L^0(\Omega; BV(0, T; X^*)),$
 $\eta(0) = 0$ a.s.
- $s_3)$ $u(t) + \eta(t) = u_0 + \int_0^t f(s) ds + M(t), \forall t \in [0, T],$ a.s.
- $s_4)$ there exists \overline{M}_n a stochastic process satisfying (36)
such that denoting for a.s. $\omega \in \Omega$:
 $(\overline{u}_n(\omega, \cdot), \overline{\eta}_n(\omega, \cdot)) = GD(A; u_0(\omega), f(\omega, \cdot), \overline{M}_n(\omega, \cdot))$
then $\overline{u}_n \rightarrow u, \overline{\eta}_n \rightarrow \eta$ in $L_{ad}^0(\Omega; C([0, T]; H))$ as $n \rightarrow \infty$ and
 $\sup_n \mathbb{E} \|\overline{\eta}_n\|_{BV([0, T]; X^*)} < +\infty.$

Remark that the adaptability of the stochastic processes u and $\eta = u_0 + \int_0^t f(s) ds + M - u$ is obtained from $s_4)$ and the continuity of the Skorohod mapping

$$(u_0(\omega), f(\omega, \cdot), \overline{M}_n(\omega, \cdot)) \xrightarrow{\Gamma} \overline{u}_n(\omega, \cdot)$$

(see Theorem 4). We have not confusion if we denote, also $u = GS(A; u_0, f, M)$ since η uniquely defined by $s_3)$.

Proposition 13 a) If $(u, \eta) = GS(A; u_0, f, M)$ and $(v, \eta) = GS(A; v_0, g, N)$ are two generalized solution of the Cauchy problem (35) then

$$\begin{aligned} |u(t) - v(t)|^2 &\leq |u(s) - v(s)|^2 + 2\alpha \int_s^t |u - v|^2 d\tau \\ &+ 2 \int_s^t (u - v, f - g) d\tau + 2 \int_s^t (u(\tau) - v(\tau), dM(\tau) - dN(\tau)) \\ &+ \langle M - N \rangle(t) - \langle M - N \rangle(s), \end{aligned} \quad (38)$$

for all $0 \leq s \leq t \leq T$, a.s. $\omega \in \Omega$.

b) The equation (35) has at most one generalized solution.

Proof. From Definition 12 and the inequality (21) we have for all $0 \leq s \leq t \leq T$:

$$\begin{aligned} &|\overline{u}_n(\omega, t) - \overline{M}_n(\omega, t) - \overline{v}_n(\omega, t) - \overline{N}_n(\omega, t)|^2 \leq \\ &\leq |\overline{u}_n(\omega, s) - \overline{M}_n(\omega, s) - \overline{v}_n(\omega, s) + \overline{N}_n(\omega, s)|^2 + 2\alpha \int_s^t |\overline{u}_n - \overline{v}_n|^2 ds \\ &+ 2 \int_s^t (\overline{u}_n(\omega, \tau) - \overline{M}_n(\omega, \tau) - \overline{v}_n(\omega, \tau) + \overline{N}_n(\omega, \tau), f(\omega, \tau) - g(\omega, \tau)) d\tau \\ &+ 2 \int_s^t \langle \overline{M}_n(\omega, \tau) - \overline{N}_n(\omega, \tau), d\overline{\eta}_n(\omega, \tau) - d\overline{\zeta}_n(\omega, \tau) \rangle \text{ a.s. } \omega \in \Omega. \end{aligned}$$

The continuity with respect to t and s involved that this inequality holds for all $0 \leq s \leq t \leq T$ and $\omega \in \Omega$, $\mathbb{P}(\Omega_0) = 1$. Using Proposition 10 (equality 30) with $u = \bar{u}_n - \bar{v}_n$, $m = \bar{M}_n - \bar{N}_n$, $f := f - g$, $\eta = \bar{\eta}_n - \bar{\zeta}_n$ from this last inequality we have

$$\begin{aligned} |\bar{u}_n(t) - \bar{v}_n(t)|^2 &\leq |\bar{u}_n(s) - \bar{v}_n(s)|^2 + 2\alpha \int_s^t |\bar{u}_n - \bar{v}_n|^2 d\tau + \\ &+ 2 \int_s^t (\bar{u}_n - \bar{v}_n, f - g) d\tau + \langle \bar{M}_n - \bar{N}_n \rangle(t) - \langle \bar{M}_n - \bar{N}_n \rangle(s) \\ &+ 2 \int_s^t (\bar{u}_n(\tau) - \bar{v}_n(\tau), d\bar{M}_n(\tau) - d\bar{N}_n(\tau)), \end{aligned} \quad (39)$$

for all $0 \leq s \leq t \leq T$; a.s. $\omega \in \Omega$, which yields (38) passing to limit as $n \rightarrow \infty$. The uniqueness is, obviously, the consequence of (38). \blacksquare

Theorem 14 *Under the assumptions (A_1) and (A_2) the initial problem (35) has a unique generalized (stochastic) solution (u, η) , $(u, \eta) = GS(A; u_0, f, M)$. Moreover this solution satisfies:*

$$\begin{aligned} a) \quad &u \in L_{ad}^2(\Omega; C([0, T]; H)) \\ b) \quad &\eta \in L_{ad}^2(\Omega; C([0, T]; H)) \cap L^1(\Omega; BV(0, T; X^*)) \\ c) \quad &\mathbb{E} \sup_{t \in [0, T]} |u(t)|^2 + \mathbb{E} \|\eta\|_{BV([0, T]; X^*)} \leq C_0[1 + \mathbb{E}|u_0|^2 + \\ &+ \mathbb{E} \left(\int_0^T |f(s)| ds \right)^2 + \mathbb{E}|M(T)|^2] \quad (\S) \end{aligned} \quad (40)$$

and if $(u, \eta) = GS(A; u_0, f, M)$, $(v, \zeta) = GS(A; v_0, g, N)$ then for all $t \in [0, T]$

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, t]} |u(s) - v(s)|^2 &\leq C(\alpha, T) [\mathbb{E}|u_0 - v_0|^2 + \mathbb{E} \left(\int_0^t |f - g| ds \right)^2 \\ &+ \mathbb{E}|M(t) - N(t)|^2] \end{aligned} \quad (41)$$

Proof. The inequality (41) is obtained easily from (38). Let $[x, y] \in A$. Then $v_n(t) = x$, $\zeta_n(t) = yt$ is a generalized (determinist) solution corresponding to $u_0 = x$, $f(t) = y$ and $M = 0$. Let $\tau_{n,R}(\omega) = \inf \{t \in [0, T] : |\bar{u}_n(\omega, t)| \geq R\}$, and $\tau_{n,R}(\omega) = T$ if the set from inf is empty; $\tau_{n,R}$ is a stopping time since $\bar{u}_n \in L_{ad}^0(\Omega; C([0, T]; H))$. Substituting in (39) $t = \tau_{n,R}(\omega)$ and using the properties of the stochastic integral, by elementary calculus we obtain:

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} |\bar{u}_n(t \wedge \tau_{n,R}) - x|^2 &\leq C(\alpha, T) [\mathbb{E}|u_0 - x|^2 + \mathbb{E}\|f - y\|_{L^1(0, T; H)}^2 \\ &+ \mathbb{E}|\bar{M}_n(T)|^2], \quad \forall [x, y] \in A. \end{aligned}$$

Which implies for $R \nearrow +\infty$ that

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} |\bar{u}_n(t) - x|^2 &\leq C(\alpha, T) [\mathbb{E}|u_0 - x|^2 + \mathbb{E}\|f - y\|_{L^1(0, T; H)}^2 \\ &+ \mathbb{E}|\bar{M}_n(T)|^2], \quad \forall [x, y] \in A. \end{aligned} \quad (42)$$

$\S C_0 = C_0(T, r_0, h_0, a_1, a_2) > 0$

Hence $\bar{u}_n \in L_{ad}^2(\Omega; C([0, T]; H))$ and also

$$\bar{\eta}_n = u_0 + \int_0^\cdot f ds + \bar{M}_n - \bar{u}_n \in L_{ad}^2(\Omega; C([0, T]; H))$$

Now from (39) for $v_n = u_{n+k}$ we have

$$\begin{aligned} |\bar{u}_n(t) - \bar{u}_{n+k}(t)|^2 &\leq 2|\alpha| \int_0^t |\bar{u}_n(s) - \bar{u}_{n+k}(s)|^2 ds + \\ &+ 2 \int_0^t (\bar{u}_n(s) - \bar{u}_{n+k}(s), d\bar{M}_n(s) - d\bar{M}_{n+k}(s)) + \langle \bar{M}_n - \bar{M}_{n+k} \rangle(t) \end{aligned}$$

for all $t \in [0, T]$, a.s. $\omega \in \Omega$, which implies by (26) and Gronwall's inequality:

$$\mathbb{E} \sup_{t \in [0, T]} |\bar{u}_n(t) - \bar{u}_{n+k}(t)|^2 \leq C(\alpha, T) \mathbb{E} |\bar{M}_n(T) - \bar{M}_{n+k}(T)|^2.$$

Hence $\exists u, \eta \in L_{ad}^2(\Omega; C([0, T]; H))$ such that as $n \rightarrow \infty$:

$$\begin{aligned} \bar{u}_n &\rightarrow u \\ \bar{\eta}_n &= u_0 + \int_0^\cdot f ds + \bar{M}_n - \bar{u}_n \rightarrow \eta = u_0 + \int_0^\cdot f + M - u \\ &\text{in } L_{ad}^2(\Omega; C([0, T]; H)) \text{ and} \\ \mathbb{E} \sup_{t \in [0, T]} |\bar{u}_n(t) - u(t)|^2 &\leq C(\alpha, T) \mathbb{E} |\bar{M}_n(T) - M(T)|^2 \end{aligned} \tag{43}$$

and

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in [0, T]} |u(t) - x|^2 \right) &\leq C(\alpha, T) [\mathbb{E} |u_0 - x|^2 + \mathbb{E} \|f - y\|_{L^1(0, T; H)}^2 + \\ &+ \mathbb{E} |M(T)|^2] \quad \forall [x, y] \in A \end{aligned} \tag{44}$$

From (38), (39) follows by a standard calculus (using (26-c) and Gronwall's inequality). Of course $u(t) \in \overline{D(A)}$, $\forall t \in [0, T]$, a.s. $u(0) = u_0$, $\eta(0) = 0$ since $\bar{u}_n, \bar{\eta}_n$ satisfy these conditions.

From Remark 7 we have a.s. $\omega \in \Omega$:

$$\begin{aligned} &|\bar{u}_n(t) - \bar{M}_n(t) - h_0|^2 + 2r_0 \|\bar{\eta}_n\|_{BV([0, T]; X^*)} \leq |u_0 - h_0|^2 + \\ &+ 2a_1 \int_0^t |\bar{u}_n(s)|^2 ds + 2a_2 t + 2 \int_0^t (f(s), \bar{u}_n(s) - \bar{M}_n(s) - h_0) ds + \\ &+ 2 \int_0^t (\bar{M}_n(s), d\bar{\eta}_n(s)) \end{aligned}$$

But by Proposition 10-a)

$$\begin{aligned}
2 \int_0^t (\overline{M}_n(s), d\overline{\eta}_n(s)) ds &= 2 (\overline{M}_n(t), \overline{\eta}_n(t)) - 2 \int_0^t (\overline{\eta}_n(s), d\overline{M}_n(s)) \\
&= 2 \left(\overline{M}_n(t), u_0 + \int_0^t f(s) ds + \overline{M}_n(t) - \overline{u}_n(t) \right) - 2 \int_0^t (u_0 + \int_0^s f(\tau) d\tau + \\
&\quad + \overline{M}_n(s) - u_n(s), d\overline{M}_n(s)) \\
&= 2 \int_0^t (f(s), \overline{M}_n(s)) ds + |\overline{M}_n(t)|^2 + \langle \overline{M}_n \rangle(t) - 2 (\overline{M}_n(t), \overline{u}_n(t)) \\
&\quad + 2 \int_0^t (u_n(s), d\overline{M}_n(s)).
\end{aligned}$$

Hence

$$\begin{aligned}
|\overline{u}_n(t)|^2 + 2r_0 \|\overline{\eta}_n\|_{BV([0,t];X^*)} &\leq 2|u_0|^2 + 10|h_0|^2 + 2|a_2|T + \\
+ 9 \left(\int_0^T |f(s)| ds \right)^2 + \langle \overline{M}_n \rangle(T) + \frac{1}{4} \sup_{s \in [0,t]} |\overline{u}_n(s)|^2 + 2|a_1| \int_0^t |\overline{u}_n(s)|^2 ds + \\
+ 2 \sup_{s \in [0,t]} \left| \int_0^s (\overline{u}_n(\tau), d\overline{M}_n(\tau)) d\tau \right|
\end{aligned}$$

and by (26-c) we conclude: there exists a positive constant $C_0 = C_0(T, r_0, h_0, a_1, a_2)$ such that

$$\begin{aligned}
\mathbb{E} \sup_{t \in [0,T]} |\overline{u}_n(t)|^2 + \mathbb{E}(\|\overline{\eta}_n\|_{BV([0,T];X^*)}) &\leq \\
\leq C_0[1 + \mathbb{E}|u_0|^2 + \mathbb{E}\|f\|_{L^1(0,T;H)}^2 + \mathbb{E}|\overline{M}_n(T)|] &\quad (45)
\end{aligned}$$

Finally the inequality (45) and the following lemma complete the proof of Theorem 14. ■

Lemma 15 *If*

$$\begin{aligned}
g, g_k &\in L^1(\Omega; C([0, T]; H)), \quad k \in \mathbb{N}^* \\
\mathbb{E} \|g_k\|_{BV([0,T];X^*)} &\leq D \equiv \text{const}, \quad \forall k \in \mathbb{N}^* \\
g_k &\rightarrow g \text{ in } C([0, T]; H), \text{ a.s. } \omega \in \Omega,
\end{aligned}$$

then

$$\begin{aligned}
g &\in L^1(\Omega; BV([0, T]; X^*)), \text{ and} \\
\mathbb{E} \|g\|_{BV([0,T];X^*)} &\leq D.
\end{aligned}$$

Proof. Let $\Delta_N : 0 = t_0^{(N)} < t_1^{(N)} < \dots < t_{k_N}^{(N)} = T$ with

$$\begin{aligned} \nu(\Delta_N) &= \max_i \left| t_{i+1}^{(N)} - t_i^{(N)} \right| \rightarrow 0, \text{ as } N \rightarrow \infty \\ S(g; \Delta_N) &= \sum_{i=0}^{k_N-1} \left\| g(t_{i+1}^{(N)}) - g(t_i^{(N)}) \right\|_{X^*} \nearrow \|g\|_{BV([0,T];X^*)} \end{aligned}$$

Then we have

$$\mathbb{E} S(g_k; \Delta_N) \leq \mathbb{E} \|g_k\|_{BV([0,T];X^*)} \leq D, \quad \forall k \in \mathbb{N}^*,$$

Passing to $\liminf_{k \rightarrow \infty}$, by Fatou Lemma, we obtain $\mathbb{E} S(g; \Delta_N) \leq D$.

Now the monotone convergence theorem (Beppv-Lévy Lemma) as $N \rightarrow \infty$ yields:

$$\mathbb{E} \|g\|_{BV([0,T];X^*)} \leq D.$$

■

Corollary 16 *Under the assumptions of Theorem 14 if $(u_\varepsilon, \eta_\varepsilon)$ is the solution of the approximating problem*

$$\begin{aligned} du_\varepsilon(t) + (A_\varepsilon^\alpha u_\varepsilon(t) - \alpha u_\varepsilon(t)) dt &= f(t) dt + dM(t) \\ u_\varepsilon(0) &= u_0 \\ \eta_\varepsilon(t) &= \int_0^t (A_\varepsilon^\alpha(u_\varepsilon(s)) - \alpha u_\varepsilon(s)) ds \end{aligned} \tag{46}$$

where A_ε^α is the Yosida approximation of the maximal monotone operator $A + \alpha I$ and $0 < \varepsilon < \frac{1}{|\alpha|+1}$, then there exists a constant $C_0 = C_0(T, r_0, h_0, a_1, a_2) > 0$ such that

$$\begin{aligned} \mathbb{E} \sup_{t \in [0,T]} |u_\varepsilon(t)|^2 + \mathbb{E} \|\eta_\varepsilon\|_{BV([0,T];X^*)} &\leq C_0(1 + \mathbb{E} |u_0|^2 + \mathbb{E} \|f\|_{L^1(0,T;H)} + \\ &\quad + \mathbb{E} |M(T)|^2) \end{aligned} \tag{47}$$

and $\lim_{\varepsilon \searrow 0} u_\varepsilon = u$, $\lim_{\varepsilon \searrow 0} \eta_\varepsilon = \eta$ in $L_{ad}^2(\Omega; C([0,T]; H))$.

Proof. As we could see in the proof of Corollary 6

$$r_0 \|(A_\varepsilon^\alpha - \alpha I)x\|_{X^*} \leq ((A_\varepsilon^\alpha - \alpha I)x, x - h_0) + b_1 |x|^2 + b_2 \tag{48}$$

with $b_i = b_i(\alpha, h_0, a_1, a_2) > 0$. By Energy Equality for $u_\varepsilon - h_0$ we have:

$$\begin{aligned} |u_\varepsilon(t) - h_0|^2 + 2 \int_0^t (u_\varepsilon(s) - h_0, d\eta_\varepsilon(s)) &= |u_0 - h_0|^2 + \\ + 2 \int_0^t (f(s), u_\varepsilon(s) - h_0) ds + 2 \int_0^t (u_\varepsilon(s) - h_0, dM(s)) &+ \langle M \rangle(t) \end{aligned}$$

which implies, by (48) and a standard calculus the inequality (47). If we approximate the martingale M by \overline{M}_n as in Theorem 14 then for the corresponding solution $\overline{u}_{\varepsilon n}, \overline{\eta}_{\varepsilon n}$ of (47)

the inequality (47) is fulfilled . We know, by Corollary 6, that for every n fixed $y_{\varepsilon,n}(\omega) = \sup_{t \in [0,T]} |\bar{u}_{\varepsilon,n}(\omega, t) - \bar{u}_n(\omega, t)|$ satisfies $\lim_{\varepsilon \searrow 0} y_{\varepsilon,n}(\omega) = 0$ a.s. $\omega \in \Omega$, and by Proposition 1 and the proof of Corollary 6 $|y_{\varepsilon,n}(\omega)|^2 \leq D_n(\omega)$, $\mathbb{E} D_n < \infty$. Hence $\exists \lim_{\varepsilon \searrow 0} u_{\varepsilon,n} = \bar{u}_n$ in $L_{ad}^2(\Omega; C([0, T]; H))$. Also by Ito's formula we have

$$\begin{aligned} |u_{\varepsilon}(t) - \bar{u}_{\varepsilon,n}(t)|^2 &\leq 2|\alpha| \int_0^t |u_{\varepsilon}(s) - \bar{u}_{\varepsilon,n}(s)|^2 ds + \\ &+ 2 \int_0^t (u_{\varepsilon}(s) - \bar{u}_{\varepsilon,n}(s), dM(s) - \overline{M}_n(s)) + \langle M - \overline{M}_n \rangle(t), \end{aligned}$$

which implies

$$\mathbb{E} \sup_{t \in [0,T]} |u_{\varepsilon}(t) - \bar{u}_{\varepsilon,n}(t)|^2 \leq C(\alpha, T) \mathbb{E} |M(T) - \overline{M}_n(T)|^2$$

Finally since

$$|u_{\varepsilon} - u|^2 \leq 3 \left(|u_{\varepsilon} - \bar{u}_{\varepsilon,n}|^2 + |\bar{u}_{\varepsilon,n} - \bar{u}_n|^2 + |\bar{u}_n - u|^2 \right)$$

then

$$\limsup_{\varepsilon \searrow 0} \mathbb{E} \sup_{t \in [0,T]} |u_{\varepsilon}(t) - u(t)|^2 \leq C_1(r, \alpha, T) \left[\mathbb{E} |M(t) - \overline{M}_n(t)|^2 \right]$$

for all $n \in N^*$, that is $\exists \lim_{\varepsilon \searrow 0} u_{\varepsilon} = u$ and $\exists \lim_{\varepsilon \searrow 0} \eta_{\varepsilon} = \lim_{\varepsilon \searrow 0} (u_0 + \int_0^\cdot f ds + M - u_{\varepsilon}) = \eta$ in $L_{ad}^2(\Omega; C([0, T]; H))$. \blacksquare

3.3 Monotone SDE with state depending diffusion

We shall work in the context of the spaces $X \subset H \subset X^*$ introduced in Subsection 2.1, and in the context of the stochastic elements defined in Subsection 3.1 as the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$, the U -Hilbert valued Wiener process $\{W(t), t \geq 0\}$ with the covariance operator $Q \in L(U, U)$, the stochastic integral etc. Consider the multivalued stochastic differential equation:

$$\begin{cases} du(t) + Au(t) dt \ni f(t, u(t)) dt + B(t, u(t)) dW(t) \\ u(0) = u_0, \quad t \in [0, T], \end{cases} \quad (49)$$

where we put the assumptions:

$$\begin{aligned} & \left(\tilde{H}_0 \right) \quad u_0 \in L^2 \left(\Omega, \mathcal{F}_0, \mathbb{P}; \overline{D(A)} \right) \\ & \left(\tilde{H}_1 \right) \begin{cases} i) \quad A : H \rightarrow 2^H \text{ is a maximal monotone operator,} \\ ii) \quad \exists r_0, a_1, a_2 > 0, \exists h_0 \in H \text{ such that} \\ \quad r_0 \|y\|_* \leq (y, x - h_0) + a_1 |x|^2 + a_2 \text{ for all } \forall [x, y] \in A \quad (\P) \end{cases} \end{aligned}$$

$$\begin{aligned}
& \left(\tilde{H}_2 \right) \left\{ \begin{array}{l} i) f : \Omega \times [0, T] \times H \mapsto H \text{ is progressively measurable i.e.} \\ \quad \forall t \in [0, T], \quad f \cdot 1_{[0, t]} \text{ is } (\mathcal{F}_t \otimes \mathcal{B}_{[0, t]} \otimes \mathcal{B}_H, \mathcal{B}_H)\text{-measurable,} \\ ii) \exists L_1, b_1 > 0 \text{ such that a.s. } \omega \in \Omega : \\ \quad |f(t, u) - f(t, v)| \leq L_1 |u - v|, \\ \quad |f(t, v)|^2 \leq b_1 (1 + |u|^2), \text{ for all } u, v \in H, \text{ a.e. } t \in]0, T[. \end{array} \right. \\
& \left(\tilde{H}_3 \right) \left\{ \begin{array}{l} i) B : \Omega \times [0, T] \times H \rightarrow L^2(U_0, H) \text{ is progressively measurable,} \\ ii) \exists L, b > 0 \text{ such that a.s. } \omega \in \Omega : \\ \quad |B(t, u) - B(t, v)|_Q^2 \leq L |u - v|^2 \\ \quad |B(t, u)|_Q^2 \leq b (1 + |u|^2) \quad \text{for all } u, v \in H, \text{ a.e. } t \in]0, T[\end{array} \right.
\end{aligned}$$

We shall try to find a solution $u \in L_{ad}^2(\Omega; C([0, T]; H))$ for the equation (49). First we remark that for a such stochastic process u we have:

$$\begin{aligned}
f(\cdot, u(\cdot)) &\in L_{ad}^2(\Omega \times]0, T[; H), \\
B(\cdot, u(\cdot)) &\in L_{ad}^2(\Omega \times]0, T[; \mathcal{L}^2(U_0, H))
\end{aligned}$$

and if we denote

$$F(t; u) = \int_0^t f(s, u(s)) ds \quad \text{and} \quad M(t; u) = \int_0^t B(s, u(s)) dW(s)$$

then

$$F(\cdot; u) \in L_{ad}^2(\Omega; C([0, T]; H)) \quad \text{and} \quad M(\cdot; u) \in M^2(0, T; H).$$

Definition 17 A stochastic process u is a (generalized) solution of multivalued SDE (49) if

$$\begin{aligned}
a) \quad & u \in L_{ad}^2(\Omega; C([0, T]; H)), \quad u(0) = u_0, \\
& u(t) \in \overline{D(A)}, \quad \forall t \in [0, T], \quad \text{a.s. } \omega \in \Omega, \\
b) \quad & \eta = u_0 + F(\cdot, u) + M(\cdot, u) - u \in L^1(\Omega; BV([0, T]; X^*)) \\
c) \quad & \mathbb{E} |u(t) - z(t)|^2 \leq \mathbb{E} |u(s) - z(s)|^2 + \\
& + 2\mathbb{E} \int_s^t (f(\tau, u) - g(\tau), u(\tau) - z(\tau)) d\tau + \mathbb{E} \int_s^t |B(\tau, u) - D(\tau)|_Q^2 d\tau, \\
& \text{for all } 0 \leq s \leq t \leq T, \quad \forall g \in L_{ad}^2(\Omega \times]0, T[; H), \\
& \forall D \in L_{ad}^2(\Omega \times]0, T[; \mathcal{L}^2(U_0, H)), \quad \forall z \in L_{ad}^2(\Omega; C([0, T]; \overline{D(A)})) \\
& \text{such that } z = GS(A; z(0), g, \int_0^\cdot D(s) dW(s)).
\end{aligned} \tag{50}$$

The existence and uniqueness result will be obtained from the following two propositions

[¶]see Theorem 2.3

Proposition 18 *Let $(\tilde{H}_0, \tilde{H}_3)$ be satisfied. Then the following problem has a unique solution:*

$$\begin{aligned}
a) \quad & u \in L_{ad}^2(\Omega; C([0, T]; H)), \quad u(0) = u_0, \\
& u(t) \in \overline{D(A)}, \quad \forall t \in [0, T], \quad \text{a.s. } \omega \in \Omega, \\
b) \quad & \eta = u_0 + F(\cdot; u) + M(\cdot; u) - u \in L^1(\Omega; BV([0, T]; X^*)), \\
c) \quad & u = GS(u_0, f(\cdot, u), M(\cdot; u)).
\end{aligned} \tag{51}$$

Proof. Under the assumptions $(\tilde{H}_0, \tilde{H}_3)$ it follows that for every v from the space $L_{ad}^2(\Omega; C([0, T]; H))$ the triplet $(u_0, f(\cdot, v), M(\cdot; v))$ satisfies the hypotheses (A_2) of Theorem 14. Hence there exists a corresponding unique generalized solution $(\bar{v}, \bar{\eta})$. This solution satisfies:

$$\begin{aligned}
\bar{v} &\in L_{ad}^2(\Omega; C([0, T]; H)), \quad \bar{v}(0) = u_0, \\
\bar{v}(t) &\in \overline{D(A)}, \quad \forall t \in [0, T], \quad \text{a.s. } \omega \in \Omega, \\
\bar{\eta} &= u_0 + F(\cdot; v) + M(\cdot; v) - \bar{u} \in L_{ad}^2(\Omega; C([0, T]; H)) \cap \\
&\quad L^1(\Omega; BV([0, T]; X^*)).
\end{aligned}$$

Denote the mapping $\Lambda : L_{ad}^2(\Omega; C([0, T]; H)) \rightarrow L_{ad}^2(\Omega; C([0, T]; H))$, $\Lambda(v) = \bar{v}$. We show that Λ has a unique fix point. Let $a > 0$. For $v \in L_{ad}^2(\Omega; C([0, T]; H))$ denote

$\|v\|_t = \left(\mathbb{E} \sup_{s \in [0, t]} |v(s)|^2 \right)^{1/2}$ and $\|v\|_a = \sup_{t \in [0, T]} (e^{-at} \|v\|_t)$. The usual norm, $\|\cdot\|_T$, on $L_{ad}^2(\Omega; C([0, T]; H))$ is equivalent to $\|\cdot\|_a$ since $e^{-aT} \|v\|_T \leq \|v\|_a \leq \|v\|_T$. From Theorem 14 (the inequality (41)) we have:

$$\begin{aligned}
\|\bar{v}_1 - \bar{v}_2\|_t^2 &\leq C(\alpha, T) \left[\mathbb{E} \int_0^t |f(v_1) - f(v_2)|^2 ds \right. \\
&\quad \left. + \mathbb{E} \int_0^t |B(v_1) - B(v_2)|_Q^2 ds \right] \\
&\leq C(L_1^2 + L) \int_0^t \|v_1 - v_2\|_s^2 ds \\
&= C_1 \int_0^t e^{-2as} \|v_1 - v_2\|_s^2 e^{2as} ds \\
&\leq C_1 \|v_1 - v_2\|_a^2 \frac{e^{2at} - 1}{2a}
\end{aligned}$$

which gets, as $a \geq 2(C_1 + 1)$:

$$\|\bar{v}_1 - \bar{v}_2\|_a \leq \frac{1}{2} \|v_1 - v_2\|_a$$

that is Λ is a contraction mapping in $L_{ad}^2(\Omega; C([0, T]; H))$. By Banach fixed point theorem a unique $v \in L_{ad}^2(\Omega; C([0, T]; H))$ exists such that $\Lambda v = v$. Hence the problem (51) has a unique solution. \blacksquare

Proposition 19 *Let the hypotheses $(\tilde{H}_0, \dots, \tilde{H}_3)$ be satisfied. Then u is the solution of the equation (49) in the sense of Definition 17 if and only if u is solution of the problem (51).*

Proof. Let u be a stochastic process satisfying (50). Then (under the assumptions $(\tilde{H}_0, \dots, \tilde{H}_3)$, by Theorem 14 there exists a unique generalized solution $(\bar{u}, \bar{\eta})$ corresponding to $(u_0, f(\cdot, u(\cdot)), M(\cdot; u))$, $\bar{u} = GS(A; u_0, f(\cdot, u), M(\cdot; u))$, and this solution satisfies (51-a,b). If we put in (50): $s = 0$, $z(0) = u_0$, $g = f(\cdot, u(\cdot))$, $D = B(\cdot, u(\cdot))$, $z = \bar{u}$ we have $\mathbb{E} |u(t) - \bar{u}(t)|^2 \leq 0$, which yields, by the continuity of the trajectories, that $u = \bar{u}$ in $L_{ad}^2(\Omega; C([0, T]; H))$. Hence u is solution of (51).

Converse if u is solution of (51), then u satisfies the conditions a) and b) of Definition 17 and, also, the condition c) via Proposition 13. \blacksquare

Propositions 18 and 19 yield clearly:

Theorem 20 *Under the assumptions $(\tilde{H}_0, \dots, \tilde{H}_3)$ the equation (49) has a unique solution in the sense of Definition 17. Moreover if $u_{01}, u_{02} \in L_{ad}^{2p}(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, $p \in [1, \infty)$, then $u_1 = u(\cdot; u_{01})$, $u_2 = u(\cdot; u_{02}) \in L_{ad}^{2p}(\Omega; C([0, T]; H))$, and*

$$\begin{cases} a) \mathbb{E} \sup_{t \in [0, T]} |u(t; u_{01})|^{2p} \leq C_1(1 + \mathbb{E} |u_{01}|^{2p}), \\ b) \mathbb{E} \sup_{t \in [0, T]} |u(t; u_{01}) - u(t; u_{02})|^{2p} \leq C_2 \mathbb{E} |u_{01} - u_{02}|^{2p}, \end{cases} \quad (52)$$

where $C_1 = C_1(b, b_1, p, T, x, y) > 0$ and $C_2 = C_2(L_1, L, T, p) > 0$, $[x, y] \in A$ arbitrary fixed.

Proof. Now we have to do is proving (52). By Propositions 18 and 13 for all $[x, y] \in A$ we have:

$$\begin{aligned} |u(t) - x|^2 &\leq |u_0 - x|^2 + 2 \int_0^t (u - x, f(u) - y) ds \\ &+ \int_0^t |B(u) - 0|_Q^2 ds + 2 \int_0^t (u - x, B(u) - 0) dW(s) \end{aligned} \quad (53)$$

$\forall t \in [0, T]$, a.s. $\omega \in \Omega$, since $x = GS(x, y, \int_0^\cdot 0 dW)$.

Let the stopping time:

$$\tau_n(\omega) = \begin{cases} \inf\{t \in [0, T] : |u(\omega, t)| \geq n\}, \\ T, \quad \text{if } |u(\omega, t)| < n, \forall t \in [0, T]. \end{cases}$$

We substitute in (53) t by $\tau_n(\omega) \wedge t$. Using Burkholder-Davis-Gundy inequality we have:

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, t]} \left| \int_0^{s \wedge \tau_n} (u - x, B(u) dW) \right|^p &\leq \\ &\leq 9(2p)^p \mathbb{E} \left(\int_0^{t \wedge \tau_n} |u - x|^2 |B(u)|_Q^2 ds \right)^{p/2} \\ &\leq \frac{1}{2} \mathbb{E} \sup_{s \in [0, t]} |u(s \wedge \tau_n)|^{2p} + C_1 \left(1 + \mathbb{E} \int_0^{t \wedge \tau_n} |u(s)|^{2p} ds \right), \end{aligned}$$

where $C_1 = C_1(b, T, p, x) > 0$ is independent of n . Then from (53), after some elementary calculus, we obtain:

$$\mathbb{E} \sup_{s \in [0, t]} |u(s \wedge \tau_n)|^{2p} \leq C \left(1 + \mathbb{E} |u_0|^{2p} + \mathbb{E} \int_0^{t \wedge \tau_n} |u(s)|^{2p} ds \right)$$

which yields:

$$\mathbb{E} \sup_{s \in [0, t]} |u(s \wedge \tau_n)|^{2p} \leq C \left(1 + \mathbb{E} |u_0|^{2p}\right)$$

and passing to limit as $n \rightarrow \infty$, (52-a) follows. The inequality (52-b) is obtained in the same manner. From Propositions 18 and 13 we have

$$\begin{aligned} |u_1(t) - u_2(t)|^2 &\leq |u_{01} - u_{02}|^2 + 2 \int_0^t (u_1 - u_2, f(u_1) - f(u_2)) ds \\ &\quad + \int_0^t |B(u_1) - B(u_2)|_Q^2 ds + 2 \int_0^t (u_1 - u_2, B(u_1) - B(u_2)) dW(s) \end{aligned}$$

As above using Burkholder-Davis-Gundy inequality and Gronwall inequality it is clear that this last inequality yields (52-b). \blacksquare

Corollary 21 *Let the assumptions $(\tilde{H}_0, \dots, \tilde{H}_3)$ be satisfied, $p \in [1, \infty)$ and $u_0 \in L^p(\Omega, F_0, \mathbb{P}; H)$. If $(u_\varepsilon, \eta_\varepsilon)$ is the solution of the approximating problem:*

$$\begin{aligned} du_\varepsilon(t) + A_\varepsilon(u_\varepsilon(t)) dt &= f(t, u_\varepsilon(t)) dt + B(t, u_\varepsilon(t)) dW(t) \\ u_\varepsilon(0) &= u_0 \\ \eta_\varepsilon(t) &= \int_0^t A_\varepsilon(u_\varepsilon(s)) ds, \end{aligned} \tag{54}$$

$\varepsilon \in (0, 1]$, where A_ε is the Yosida approximation of the maximal monotone operator A . Then there exists a positive constant $C = C(T, r_0, h_0, a_1, a_2, b_1, b)$ such that:

$$\mathbb{E} \sup_{t \in [0, T]} |u_\varepsilon(t)|^{2p} + \mathbb{E} \|\eta_\varepsilon\|_{BV([0, T]; X^*)}^p \leq C(1 + \mathbb{E} |u_0|^{2p}) \tag{55}$$

and

$$\lim_{\varepsilon \searrow 0} u_\varepsilon = u, \quad \lim_{\varepsilon \searrow 0} \eta_\varepsilon = \eta \quad \text{in } L_{ad}^2(\Omega; C([0, T]; H)) \tag{56}$$

Proof. We know, by Pardoux result (Theorem 11) or directly by a fixed point argument, that, under the assumptions $(\tilde{H}_0, \dots, \tilde{H}_3)$, the equation (54) has a unique solution $u_\varepsilon \in L_{ad}^2(\Omega; C([0, T]; H))$. By Energy Equality:

$$\begin{aligned} &|u_\varepsilon(t) - h_0|^2 + 2 \int_0^t (A_\varepsilon(u_\varepsilon), u_\varepsilon - h_0) ds \\ &= |u_0 - h_0|^2 + 2 \int_0^t (u_\varepsilon - h_0, f(u_\varepsilon)) ds \\ &\quad + 2 \int_0^t (u_\varepsilon - h_0, B(u_\varepsilon)) dW(s) + \int_0^t |B(u_\varepsilon)|_Q^2 ds. \end{aligned} \tag{57}$$

But by $(\tilde{H}_1 - ii)$ and (3-a,e) we have:

$$\begin{aligned} r_0 \|A_\varepsilon x\|_{X^*} &\leq (A_\varepsilon x, J_\varepsilon x - h_0) + a_1 |J_\varepsilon x|^2 + a_2 \\ &\leq (A_\varepsilon x, x - h_0) + 2a_1 |x|^2 + (8a_1 |J_1 0|^2 + a_2), \end{aligned}$$

for all $x \in H$, $\varepsilon \in (0, 1]$, and by (29-d) $\forall \delta > 0$:

$$\begin{aligned} & \mathbb{E} \sup_{s \in [0, t]} \left| \int_0^s (u_\varepsilon - h_0, B(u_\varepsilon)) dW(\tau) \right|^p \leq \\ & \leq 9(2p)^p \mathbb{E} \left(\int_0^t |u_\varepsilon - h_0|^2 |B(u_\varepsilon)|_Q^2 ds \right)^{p/2} \\ & \leq \delta \mathbb{E} \sup_{s \in [0, t]} |u(s)|^{2p} + \frac{C(p, T, h_0, b)}{\delta} \mathbb{E} \int_0^t |u_\varepsilon(s)|^{2p} ds. \end{aligned}$$

These estimates used in (57) produce clearly, by a standard calculus, (55).

From Proposition 19 we know that $u = GS(A; u_0, f(\cdot, u), M(\cdot; u))$. Hence

$$u = \lim_{\varepsilon \searrow 0} \tilde{u}_\varepsilon \text{ in } L_{ad}^2(\Omega; C([0, T]; H)),$$

where \tilde{u}_ε is the solution of the equation

$$\begin{aligned} d\tilde{u}_\varepsilon(t) + A_\varepsilon(\tilde{u}_\varepsilon(t)) dt &= f(t, u(t)) dt + B(t, u(t)) dW(t) \\ u_\varepsilon(0) &= u_0, \end{aligned}$$

$\varepsilon \in (0, 1]$.

We write the Energy Equality for $u_\varepsilon - \tilde{u}_\varepsilon$:

$$\begin{aligned} & |u_\varepsilon(t) - \tilde{u}_\varepsilon(t)|^2 + 2 \int_0^t (A_\varepsilon(u_\varepsilon) - A_\varepsilon(\tilde{u}_\varepsilon), u_\varepsilon - \tilde{u}_\varepsilon) ds \\ &= 2 \int_0^t (u_\varepsilon - \tilde{u}_\varepsilon, f(u_\varepsilon) - f(u)) ds + 2 \int_0^t (u_\varepsilon - \tilde{u}_\varepsilon, B(u_\varepsilon) - B(u)) dW(s) \\ & \quad + \int_0^t |B(u_\varepsilon) - B(u)|_Q^2 ds. \end{aligned}$$

By (\tilde{H}_2) and (\tilde{H}_3) we have

$$\begin{aligned} 2(u_\varepsilon - \tilde{u}_\varepsilon, f(u_\varepsilon) - f(u)) &\leq 3L_1 |u_\varepsilon - \tilde{u}_\varepsilon|^2 + L_1 |\tilde{u}_\varepsilon - u|^2, \\ |B(u_\varepsilon) - B(u)|_Q^2 &\leq 2L |u_\varepsilon - \tilde{u}_\varepsilon|^2 + 2L |\tilde{u}_\varepsilon - u|^2. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, t]} |u_\varepsilon(s) - \tilde{u}_\varepsilon(s)|^2 &\leq (3L_1 + 2L) \mathbb{E} \int_0^T |\tilde{u}_\varepsilon(s) - u(s)|^2 ds \\ & \quad + (3L_1 + 2L) \int_0^t \mathbb{E} \sup_{s \in [0, \tau]} |u_\varepsilon(s) - \tilde{u}_\varepsilon(s)|^2 d\tau \\ & \quad + 6 \mathbb{E} \left(\int_0^t |u_\varepsilon - \tilde{u}_\varepsilon|^2 |B(u_\varepsilon) - B(u)|_Q^2 ds \right)^{1/2} \end{aligned}$$

which easily yields

$$\mathbb{E} \sup_{s \in [0, T]} |u_\varepsilon(s) - \tilde{u}_\varepsilon(s)|^2 \leq C \mathbb{E} \sup_{s \in [0, T]} |\tilde{u}_\varepsilon(s) - u(s)|^2,$$

where $C = C(T, L, L_1) > 0$ and (56) follows. ■

4 Large time behaviour

4.1 Exponentially stability

In this section we shall study some asymptotic properties of the generalized (stochastic) solution of the equation (49). We shall consider the same context of the spaces, $X \subset H \subset X^*$, as in Section 3.3 and, the assumptions, $(\tilde{H}_0, \dots, \tilde{H}_3)$ be satisfied. Assume also that

$$\begin{aligned} \exists a \geq 0 \text{ such that } \forall [x_1, y_1], [x_2, y_2] \in A \\ (y_1 - y_2, x_1 - x_2) \geq a |x_1 - x_2|^2 \end{aligned} \quad (58)$$

Lemma 22 *If $(\tilde{H}_1 - i)$ and (58) are satisfied then $\forall \theta \in [0, 1)$, $\forall \varepsilon > 0$, such that $\varepsilon a \theta \leq 1 - \theta$, the inequality $(A_\varepsilon u - A_\varepsilon v, u - v) \geq a \theta |u - v|^2$ holds $\forall u, v \in H$.*

Proof. We have

$$\begin{aligned} (A_\varepsilon u - A_\varepsilon v, u - v) &= (A_\varepsilon u - A_\varepsilon v, J_\varepsilon u + \varepsilon A_\varepsilon u - J_\varepsilon v - \lambda A_\varepsilon v) \\ &\geq a |J_\varepsilon u - J_\varepsilon v|^2 + \lambda |A_\varepsilon u - A_\varepsilon v|^2 \\ &= a |u - v - \varepsilon (A_\varepsilon u - A_\varepsilon v)|^2 + \varepsilon |A_\varepsilon u - A_\varepsilon v|^2. \end{aligned}$$

But $|u - v|^2 \geq \theta |u|^2 - \frac{\theta}{1-\theta} |v|^2$, $\forall u, v \in H$. Hence

$$\begin{aligned} (A_\varepsilon u - A_\varepsilon v, u - v) &\geq a \theta |u - v|^2 + \varepsilon (1 - \frac{a \theta \varepsilon}{1 - \theta}) |A_\varepsilon u - A_\varepsilon v|^2 \\ &\geq a \theta |u - v|^2. \end{aligned}$$

■

Proposition 23 *Let $(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3)$ and (58) be satisfied and u, v be two generalized solutions of equation (49) corresponding to the initial dates u_0, v_0 , respectively ($u_0, v_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \overline{D(A)})$). Let $\theta \in [0, 1)$ and $\beta = 2a\theta - 2L - L_1$. Then*

$$\begin{aligned} \mathbb{E} \left(|u(t) - v(t)|^2 | \mathcal{F}_s \right) &\leq e^{-\beta(t-s)} |u(s) - v(s)|^2 \\ \text{for all } 0 \leq s \leq t \leq T, \text{ a.s. } \omega \in \Omega. \end{aligned} \quad (59)$$

Proof. By Corollary 21 we have: $u = \lim_{\varepsilon \rightarrow 0} u_\varepsilon$, $v = \lim_{\varepsilon \rightarrow 0} v_\varepsilon$ in $L_{ad}^2(\Omega; C([0, T]; H))$ where u_ε and v_ε are the solutions of the approximating equations:

$$\begin{cases} du_\varepsilon + A_\varepsilon u_\varepsilon dt = f(t, u_\varepsilon(t)) dt + B(t, u_\varepsilon(t)) dW(t) \\ u_\varepsilon(0) = u_0 \end{cases}$$

and

$$\begin{cases} dv_\varepsilon(t) + A_\varepsilon v_\varepsilon dt = f(t, v_\varepsilon(t)) dt + B(t, v_\varepsilon(t)) dW(t) \\ v_\varepsilon(0) = v_0 \end{cases}$$

respectively.

We write the Energy Equality for $|u_\varepsilon - v_\varepsilon|^2$:

$$\begin{aligned} & |u_\varepsilon(t) - v_\varepsilon(t)|^2 + 2 \int_s^t (A_\varepsilon u_\varepsilon - A_\varepsilon v_\varepsilon, u_\varepsilon - v_\varepsilon) d\tau \\ &= |u_\varepsilon(s) - v_\varepsilon(s)|^2 + 2 \int_s^t (u_\varepsilon - v_\varepsilon, f(u_\varepsilon) - f(v_\varepsilon)) ds + \\ &+ 2 \int_s^t (u_\varepsilon - v_\varepsilon, (B(u_\varepsilon) - B(v_\varepsilon)) dW) + \int_s^t |B(u_\varepsilon) - B(v_\varepsilon)|_Q^2 ds. \end{aligned}$$

Using Lemma 22 and the hypotheses (\tilde{H}_2) and (\tilde{H}_3) we have for $\varepsilon > 0$, $\varepsilon a\theta < 1 - \theta$:

$$\begin{aligned} & |u_\varepsilon(t) - v_\varepsilon(t)|^2 + 2a\theta \int_s^t |u_\varepsilon - v_\varepsilon|^2 d\tau \leq |u_\varepsilon(s) - v_\varepsilon(s)|^2 + \\ &+ (L + 2L_1) \int_s^t |u_\varepsilon - v_\varepsilon|^2 d\tau + 2 \int_s^t (u_\varepsilon - v_\varepsilon, (B(u_\varepsilon) - B(v_\varepsilon)) dW) \end{aligned}$$

for all $0 \leq s \leq t \leq T$, a.s. $\omega \in \Omega$. Let $\varepsilon = \varepsilon_n \rightarrow 0$ such that

$$u_{\varepsilon_n} \rightarrow u, \quad v_{\varepsilon_n} \rightarrow v \quad \text{in } C([0, T]; H), \quad \text{a.s. } \omega \in \Omega.$$

Passing to limit in the last inequality for $\varepsilon = \varepsilon_n \rightarrow 0$ we obtain:

$$\begin{aligned} & |u(t) - v(t)|^2 + 2a\theta \int_s^t |u - v|^2 d\tau \leq |u(s) - v(s)|^2 + \\ &+ (2L_1 + L) \int_s^t |u - v|^2 d\tau + 2 \int_s^t (u - v, (B(u) - B(v)) dW) \end{aligned}$$

for all $0 \leq s \leq t \leq T$, a.s. $\omega \in \Omega$.

Let $s_0 \in [0, T]$ be fixed, and $\varphi(\tau) = \mathbb{E}(|u(\tau) - v(\tau)|^2 | \mathcal{F}_{s_0})$. Then $\varphi(t) + \beta \int_s^t \varphi(\tau) d\tau \leq \varphi(s)$ for all $s_0 \leq s \leq t \leq T$, a.s. $\omega \in \Omega$, which implies $\varphi(t) \leq e^{-\beta(t-s)} \varphi(s)$ that is

$$\begin{aligned} \mathbb{E}(|u(t) - v(t)|^2 | \mathcal{F}_{s_0}) &\leq e^{-\beta(t-s)} \mathbb{E}(|u(s) - v(s)|^2 | \mathcal{F}_{s_0}) \\ &\text{for all } s_0 \leq s \leq t \leq T, \text{ a.s. } \omega \in \Omega \end{aligned} \tag{60}$$

The inequality (60) yields (59) for $s = s_0$. ■

Corollary 24 *Under the assumptions of Proposition 23 $\Delta(t) = e^{\beta t} |u(t) - v(t)|^2$ is a supermartingale.*

Proof. It is evidently that: $\Delta \in L_{ad}^1(\Omega; C([0, T]; R))$, $\mathbb{E}(\Delta(t) | \mathcal{F}_s) \leq \Delta(s)$ for $s \leq t$. ■

We shall consider from now on f and B that:

$$\begin{aligned} & f : \Omega \times [0, \infty[\times H \rightarrow H, \quad B : \Omega \times [0, \infty[\times H \rightarrow \mathcal{L}^2(U_0, H) \\ & \text{satisfy } (\tilde{H}_2) \text{ and } (\tilde{H}_3) \text{ respectively on each interval } [0, T] \\ & \text{with the constants } L, b_1, L_1 \text{ independent of } T \end{aligned} \tag{61}$$

If $u_{[0,T]}(t; 0, u_0)$ is the solution on $[0, T]$ then by uniqueness

$$u_{[0,T_1]}(t; 0, u_0) = u_{[0,T_2]}(t; 0, u_0), \forall t \in [0, T_1 \wedge T_2], \text{ a.s.}$$

Thus we can define $u(\cdot; 0, u_0) : \Omega \times [0, \infty[\rightarrow H$ by $u(t; 0, u_0) = u_{[0,n]}(t; 0, u_0)$ for $t \in [0, n]$, $n \in N^*$.

Theorem 25 Assume that (\tilde{H}_1) , (61) hold, and the operator A satisfies, moreover, (58) with $a > 0$. If $\beta_0 = 2a - 2L - L_1 > 0$ then for all two solutions u, v corresponding to initial dates $u_0, v_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \overline{D(A)})$, respectively, we have:

- a) $\mathbb{E} |u(t) - v(t)|^2 \leq e^{-\beta_0 t} \mathbb{E} |u_0 - v_0|^2$
- b) for all $\gamma \in [0, \beta_0)$ $\lim_{t \rightarrow \infty} e^{\gamma t} |u(t) - v(t)|^2 = 0$ in $L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ and \mathbb{P} -a.s..
- c) $\mathbb{E} \int_0^\infty |u(t) - v(t)|^2 dt \leq \frac{1}{\beta_0} \mathbb{E} |u_0 - v_0|^2$

Besides if $u_0 \equiv x \in \overline{D(A)}$ and $v_0 \equiv \tilde{x} \in \overline{D(A)}$ then for all $\gamma \in [0, \beta_0)$ there exists a random variable $\tau(\omega) = \tau(\omega; a, x - \tilde{x}) > 0$ a.s. such that

- d) $|u(\omega, t) - v(\omega, t)|^2 \leq e^{-\gamma t} |x - \tilde{x}|^2$ for all $t \geq \tau(\omega)$, a.s. $\omega \in \Omega$.

Proof. The results are easily obtained from (59) as $\theta \nearrow 1$ and $s = 0$, and Corollary 24. ■

4.2 Invariant measure

Let $D = \text{Dom}(A)$, $B_{\overline{D}}$ the σ -algebras Borel on \overline{D} , $\mathcal{M}(\overline{D})$ the space of bounded measure on $B_{\overline{D}}$, $\mathcal{M}_1^+(\overline{D})$ the space of probability measure on $B_{\overline{D}}$, $B_b(\overline{D})$ the Banach space of the bounded Borel measurable functions $g : \overline{D} \rightarrow \mathbb{R}$ with the sup-norm, and $C_b(\overline{D})$ the Banach space of the bounded continuous functions $g : \overline{D} \rightarrow \mathbb{R}$. We shall assume that (\tilde{H}_1) and (61) are satisfied. We denote $u(t; s, \xi)$ the generalized (stochastic) solution $u(t)$ of the problem

$$\begin{aligned} du(t) + (Au)(dt) &\ni f(t, u(t))dt + B(t, u(t))dW_s(t) \\ u(s) &= \xi, \quad t \in [s, \infty), \end{aligned} \tag{62}$$

where $\xi \in L^2(\Omega, \mathcal{F}_s, \mathbb{P}; \overline{D})$ and $W_s(t) = W(t) - W(s)$.

We define for $s \leq t$: $(P_{st}g)(x) = \mathbb{E}g(u(t; s, x))$, where $g \in B_b(\overline{D})$ and $x \in \overline{D}$. With a similar proof as in [8] (Theorem 9.8, Corollaries 9.9 and 9.10, p.250-252) we have:

- a) $P_{st} \in \mathcal{M}_1^+(\overline{D})$,
- b) $u(t; s, x)$ is a Markov process with transition probability P_{st} i.e.
 $\mathbb{E}(g(u(t; s, x)) | \mathcal{F}_\tau) = (P_{\tau t}g)(u(\tau; s, x))$,
 $\forall 0 \leq s \leq \tau \leq t, \forall g \in B_b(\overline{D})$,
- c) $u(t; s, x)$ has the Feller property that is $P_{st}(C_b(\overline{D})) \subset C_b(\overline{D})$,
- d) $P_{s\tau}(P_{\tau t}g) = P_{st}g, \forall g \in B_b(D), \forall 0 \leq s \leq \tau \leq t$,
- e) if $f(t, u) \equiv f(u)$, $B(t, u) \equiv B(u)$ then $P_{st} = P_{0, t-s}$

In this subsection we shall study the behaviour as $t \rightarrow \infty$ of the laws $L(u(t; u_0))$ of the random variables $u(t; u_0) = u(t; 0, u_0)$ under the assumptions

- i) $f(t, u) \equiv f(u)$, $B(t, u) \equiv B(u)$ (f and B are independent of t)
 - ii) (\tilde{H}_1) , (58) with $a > 0$, and (61) are satisfied
 - iii) $\beta_0 = a - L_1 - \frac{1}{2}L > 0$
- (64)

Let $(P_t g)(x) = \mathbb{E}g(u(t; x)) = (P_{st}g)(x)$ where $t \geq 0$, $g \in B_b(\overline{D})$. For all $\mu \in \mathcal{M}(\overline{D})$ and $g \in B_b(\overline{D})$ we define $\langle \mu, g \rangle = \int_{\overline{D}} g(x) d\mu(x)$ and the dual semigroup $P_t^* : \mathcal{M}(\overline{D}) \rightarrow \mathcal{M}(\overline{D})$ by $\langle P_t^* \mu, g \rangle = \langle \mu, P_t g \rangle$

Definition 26 A probability measure $\mu \in \mathcal{M}_1^+(\overline{D})$ is an invariant measure for the generalized (stochastic) solution $u(t; u_0)$ if $\forall t > 0 : P_t^* \mu = \mu$ or equivalent

$$\int_{\overline{D}} \mathbb{E}g(u(t; x)) d\mu(x) = \int_{\overline{D}} g(x) d\mu(x), \text{ for all } g \in B_b(\overline{D})$$

(or $\forall g \in C_b(\overline{D})$, or $\forall g = 1_\Gamma$, $\Gamma \in B_{\overline{D}}$).

As in [8] (Proposition 11.1 and 11.2, p.303-304), based only (63), we have:

Proposition 27 a) If the law of ξ is ν then the law of ξ is ν then the law of $u(t; \xi)$ is $P_t^* \nu$.
b) If $\exists \xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \overline{D})$ such that $\lim_{t \rightarrow \infty} \mathbb{E}g(u(t; \xi)) = \langle \mu, g \rangle$, $\forall g \in C_b(\overline{D})$ then μ is an invariant measure.

The first result of this subsection will give an information on the behaviour of $u(t; x)$, $x \in D(A)$ as $t \rightarrow \infty$ in the sense of $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ -convergence.

Proposition 28 Under the assumptions (64), for all $[x_0, y_0] \in A$, $\forall 0 \leq s \leq t$:

- a) $\mathbb{E}|u(t; x_0) - x_0|^2 \leq C_0 M_0$,
 - b) $\mathbb{E}|u(t; x_0) - u(s; x_0)|^2 \leq C_0 M_0 e^{-\beta_0 s} (e^{-\beta_0 s} - e^{-\beta_0 t})$
- $$\leq C_0 M_0 \beta_0 (t - s),$$
- (65)

where $C_0 = C_0(a, L, L_1)$ is a positive constant (we can put $C_0 = (\beta_0 + 2 + L) / \beta_0^2$) and

$$M_0 = |y_0|^2 + |f(x_0)|^2 + |B(x_0)|_Q^2$$
(66)

Proof. We shall use the idea from [[8]] (Theorem 11.21, p.327). Let \widetilde{W} a U -valued Q -Wiener process independent of W . We extend W for $t < 0$ by $W(t) = \widetilde{W}(-t)$, $t \leq 0$, and consider $u_{\varepsilon, \tau}(t) = u_\varepsilon(t; -\tau, x_0)$ the solution of the equation

$$\begin{cases} du_{\varepsilon, \tau}(t) + A_\varepsilon u_{\varepsilon, \tau}(t) dt = f(u_{\varepsilon, \tau}(t)) dt + B(u_{\varepsilon, \tau}(t)) dW(t) \\ u_{\varepsilon, \tau}(-\tau) = x_0, t \geq -\tau \end{cases}$$

It is easy to see that $u_{\varepsilon,\tau}(0)$ and $u_\varepsilon(\tau; 0, x_0)$ have the same law. By Energy Equality we have

$$\frac{d}{dt} \mathbb{E} |u_{\varepsilon,\tau}(t) - x_0|^2 + \mathbb{E} G_\varepsilon(u_{\varepsilon,\tau}(t)) = 0, \quad t > -\tau, \quad (67)$$

where

$$\begin{aligned} G_\varepsilon(v) &= 2(A_\varepsilon v, v - x_0) - 2(f(v), v - x_0) - |B(v)|^2 \\ &= 2(A_\varepsilon v - A_\varepsilon x_0, v - x_0) - 2(f(v) - f(x_0), v - x_0) \\ &\quad - |B(v) - B(x_0)|_Q^2 + 2(A_\varepsilon x_0, v - x_0) - 2(f(x_0), v - x_0) \\ &\quad - 2(B(v) - B(x_0), B(x_0))_Q - |B(x_0)|_Q^2 \end{aligned}$$

By Lemma 22 for $\theta \in (0, 1)$ fixed arbitrary and $\varepsilon \in (0, \frac{1-\theta}{a\theta})$ we have

$$\begin{aligned} G_\varepsilon(v) &\geq (2a\theta - 2L_1 - L - 2\delta - L\delta) |v - x_0|^2 \\ &\quad - \left(1 + \frac{1}{\delta}\right) \left(|y_0|^2 + |f(x_0)|^2 + |B(x_0)|_Q^2\right) \end{aligned}$$

for all $\delta > 0$. Using this last inequality in (67) and integrating from $-\tau$ to t we obtain for $\delta = \delta_0 = \beta_0 / (2 + L)$:

$$\mathbb{E} |u_{\varepsilon,\tau}(t) - x_0|^2 \leq \left(1 + \frac{1}{\delta_0}\right) M_0 \int_{-\tau}^t e^{(\beta_0 - 2a(1-\theta))(r-t)} dr \quad (68)$$

and more for $t = 0$:

$$\mathbb{E} |u_\varepsilon(\tau; 0, x_0) - x_0|^2 \leq \left(1 + \frac{1}{\delta_0}\right) M_0 \int_0^\tau e^{(2a(1-\theta) - \beta_0)r} dr \quad (69)$$

We pass to limit in (69) as $\varepsilon \searrow 0$ and then for $\theta \nearrow 1$; the inequality (65-a) is yielded.

Applying Ito's formula to $|u_{\varepsilon,\sigma}(t) - u_{\varepsilon,\tau}(t)|^2$, for $t \geq -\sigma \geq -\tau$ one has:

$$\frac{d}{dt} \mathbb{E} |u_{\varepsilon,\sigma}(t) - u_{\varepsilon,\tau}(t)|^2 + \mathbb{E} \tilde{G}_\varepsilon(u_{\varepsilon,\sigma}(t), u_{\varepsilon,\tau}(t)) = 0, \quad (70)$$

where

$$\begin{aligned} \tilde{G}_\varepsilon(u, v) &= 2(A_\varepsilon u - A_\varepsilon v, u - v) - 2(f(u) - f(v), u - v) - |B(u) - B(v)|_Q^2 \\ &\geq (2a\theta - 2L_1 - L) |u - v|^2 \end{aligned}$$

for $\theta \in (0, 1)$ fixed arbitrary and $\varepsilon \in (0, \frac{1-\theta}{a\theta})$. We integrate (70) from $-\sigma$ to 0 and we obtain

$$\begin{aligned} \mathbb{E} |u_\varepsilon(\sigma; 0, x_0) - u_\varepsilon(\tau; 0, x_0)|^2 &= \mathbb{E} |u_{\varepsilon,\sigma}(0) - u_{\varepsilon,\tau}(0)|^2 \\ &\leq \mathbb{E} \left(|x_0 - u_{\varepsilon,\tau}(-\sigma)|^2 \right) e^{-(2a(\theta-1) + 2\beta_0)\sigma} \\ &\leq \left(1 + \frac{1}{\delta_0}\right) M_0 e^{-\beta_0\sigma} \int_{-\tau}^{-\sigma} e^{(\beta_0 - 2a(1-\theta))r} dr \end{aligned}$$

which yields (65-b) as $\varepsilon \searrow 0$ and then $\theta \nearrow 1$. ■

Theorem 29 *Let the assumptions (64) be satisfied. Then there exists $\eta \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \overline{D})$ such that*

- $c_1)$ $\lim_{t \rightarrow \infty} u(t; u_0) = \eta$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}; H), \forall u_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \overline{D})$
- $c_2)$ $u(t; u_0)$ has a unique invariant measure $\mu = L(\eta)$

Proof. Let $[x_0, y_0] \in A$. From Proposition 28 one follows that $\exists \eta \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \overline{D})$ such that $\exists \lim_{t \rightarrow \infty} u(t; x_0) = \eta$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ and from Theorem 25 we have that $\lim_{t \rightarrow \infty} u(t; u_0) = \eta$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ for all $u_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \overline{D})$. Let $\mu = L(\eta)$ the law of η . Since the convergence in L^2 implies the convergence in law then $\lim_{t \rightarrow \infty} \mathbb{E}g(u(t; u_0)) = \mathbb{E}g(\eta) = \int_{\overline{D}} g(x) d\mu(x)$ for all $g \in C_b(\overline{D})$. Hence by Proposition 27 μ is an invariant measure. The invariant measure μ is unique since if μ_1, μ_2 are two invariant measure then $\forall t \geq 0$:

$$\langle \mu_1, g \rangle = \int_{\overline{D}} \mathbb{E}g(u(t; x)) d\mu_1(x), \text{ and } \langle \mu_2, g \rangle = \int_{\overline{D}} \mathbb{E}g(u(t; x)) d\mu_2(x),$$

which implies, as $t \rightarrow \infty$, $\langle \mu_1, g \rangle = \mathbb{E}g(\eta) = \langle \mu_2, g \rangle$ for all $g \in C_b(\overline{D})$. ■

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